1. **(24 points)** *Answer the following questions related to the shape of the graph of the function* $g(x) = x^4 - 8x^2 + 8$.

   (a) **(4 points)** *What is $g(x)$'s long term behavior as x grows very large or very negative? Describe each direction in either words or symbols.*

   As $x \to \pm\infty$, the x^4 term will be far, far larger in magnitude than either the $-8x^2$ or 8 terms. Thus the long-term behavior of this polynomial is identical to that of x^4 alone: that is, for very large x, $g(x)$ will itself be very large, and for very negative x, $g(x)$ will still be very large (and positive).

   (b) **(6 points)** *Where is $g(x)$ increasing? Where is it decreasing? Label which is which.*

   $g'(x) = 4x^3 - 16x = 4x(x-2)(x+2)$, which is zero when x is -2, 0, or 2. By probing at, for instance, $g'(-3)$, $g'(-1)$, $g'(1)$, and $g'(3)$, we shall see that $g'(x)$ is positive, and thus $g(x)$ is increasing, when $-2 < x < 0$ or $x > 2$; $g'(x)$ is negative, and thus $g(x)$ is decreasing, when $x < -2$ or $0 < x < 2$.

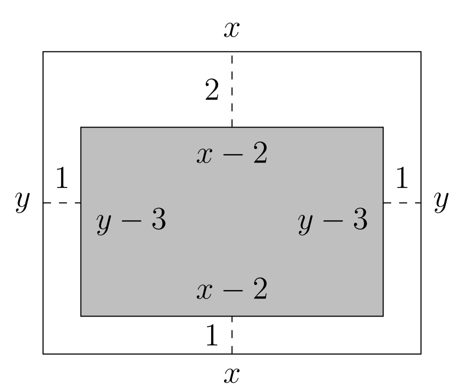
   (c) **(6 points)** *What are its critical points, and is each a local maximum, a local minimum, or neither?*

   The critical points are the zeroes identified above: $x = -2$, $x = 0$, and $x = 2$. $x = -2$ and $x = 2$ are both transition points from decrease to increase, so they are minima, while $x = 0$ is a transition point from increase to decrease, so it is a maximum.

   (d) **(8 points)** *Where is it concave up? Where is it concave down? Label which is which. Where, if anywhere, are its points of inflection?*

   $g''(x) = 12x^2 - 16$. This is zero when $x^2 = \frac{4}{3}$, or in other words at $x = \pm\frac{2}{\sqrt{3}}$. Probing the three intervals into which these points divide the number line (for instance, by calculating $g''(-2)$, $g''(0)$, and $g''(2)$), we see that $g''(x)$ is positive, so that $g(x)$ is concave up, when $x < -\frac{2}{\sqrt{3}}$ or $x > \frac{2}{\sqrt{3}}$. Likewise, $g(x)$ will be concave down for $-\frac{2}{\sqrt{3}} < x < \frac{2}{\sqrt{3}}$, and the points $x = \pm\frac{2}{\sqrt{3}}$, where the concavity transitions, will be points of inflection.

2. **(24 points)** *You have 150 square inches of paper with which to design a rectangular poster. The top margin of the poster will be 2 inches, and the bottom, left, and right margins will be 1 inch. What dimensions for the poster maximize the printable area?*



The above drawing is a representation of the scenario described; we assign the two dimensions of the sheet the labels of x and y (we could alternatively label the dimensions of the printable area with x and y, which would give correct results, but it would make the arithmetic a bit messier). Since there is a margin of 2 inches on top and 1 inch on bottom, the height of the printable area will be three inches less than the area of the sheet; likewise, the 1 inch margins on left and right will make the printed area have width of two inches less than the sheet, so the printable region is an $(x-2) \times (y-3)$ rectangle.

Our constraint in this situation is that the sheet as a whole is 150 square inches in area; thus we are constrained that the area xy is equal to 150. What we seek to maximize is the printed area, which, as seen above, has area given by the product $(x-2)(y-3)$. Our constraint may be rephrased as $y = \frac{150}{x}$, so the printable area of the sheet has area given, in terms solely of x, by the function $A(x) = (x-2)\left(\frac{150}{x} - 3\right) = 150 - 3x - \frac{300}{x} + 6$, which we seek to maximize over the entire range of possible values of x: this range of values is from 2 to 50, since if x were less than 2, there would not be enough horizontal size even for the 2 inches of margins, while if x were greater than 50, then y would be less than 3, leaving not enough vertical size even for the 3 inches of margins. Thus, our goal is to maximize the function $A(x) = 156 - 3x - \frac{300}{x}$ on the interval $[2, 50]$.

We calculate $A'(x) = -3 + \frac{300}{x^2}$, and seek the critical points of $A(x)$. Since $A'(x)$ is undefined at $x = 0$, this is a critical point; we also must determine where $A'(x) = 0$. This occurs when $3 = \frac{300}{x^2}$, or when $x^2 = 100$, which simplifies to $x = \pm 10$. Fortunately, of our three critical points, two are outside of the interval described, leaving our only maximization candidates as the values $x = 10$, $x = 2$, and $x = 50$. As might be expected, $A(2) = A(50) = 0$, since the 2×75 and 50×3 posters consist of nothing but margins, while $A(10) = 156 - 30 - 30 = 96$, so $x = 10$ is an optimal choice (or in other words, our optimal poster size is 10×15, which would have a printable area of 8×12).

3. **(18 points)** *Answer the following questions:*

   (a) **(5 points)** *Determine a region whose area is* $\lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{5}{n}\right) \arctan(2 + \frac{5i}{n})$. *You may express your answer as a definite integral, or as a description in words.*

   The area of a region under a curve $f(x)$ from $x = a$ to $x = b$ is known to be

$$\lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{b-a}{n}\right) f\left(a + \frac{b-a}{n} i\right),$$

   which bears a similarity except in details to the expression given. We must therefore find expressions to stand in for a, b, and $f(x)$ to make these expressions equivalent. Taking the most naïve decomposition of these equivalent expressions, we find that

$$\frac{b-a}{n} = \frac{5}{n}$$
$$f\left(a + \frac{b-a}{n} i\right) = \arctan\left(2 + \frac{5}{n} i\right)$$

   The first equation clearly establishes that $b - a = 5$; substituting this knowledge into the second equation, we find that our correspondences become:

$$b - a = 5$$
$$f\left(a + \frac{5}{n} i\right) = \arctan\left(2 + \frac{5}{n} i\right)$$

which lends itself to the obvious intepretation $f(x) = \arctan x$ and $a = 2$ (other interpretations are possible, and will give rise to slightly different but equally correct answers). Then, since $b - a = 5$, $b = 5 + a = 7$. Thus, the expression we were given is the area-under-a-curve formula with $a = 2$, $b = 7$, and $f(x) = \arctan x$, so this expression is the area of the region under the curve $y = \arctan x$ between $x = 2$ and $x = 7$, which might be written as $\int_2^7 \arctan x dx$.

(b) **(7 points)** *Find $f(x)$ given that $f'(x) = 16x^3 - 3x^2$ and $f(1) = 4$.*

We know that $f(x)$ is an antiderivative of $f'(x)$; the general antiderivative, worked term-by term, can be seen to be $4x^4 - x^3 + C$; thus we know that $f(x) = 4x^4 - x^3 + C$ for *some* vale of C. Plugging in the known value $f(1) = 4$, we can solve for C:

$$4 = 4 \cdot 1^4 - 1^3 + C$$
$$1 = C$$

so the specific formula for $f(x)$ is $4x^4 - x^3 + 1$.

(c) **(8 points)** *Find the general antiderivative of $h(t) = \sqrt[6]{t} + \frac{5}{t} - 2 + 4\csc^2 t - \frac{5}{1+t^2}$.*

We intepret this expression as $h(t) = t^{1/6} + \frac{5}{t} - 2 + 4\csc^2 t - \frac{5}{1+t^2}$. Using known antiderivative rules, we know that antiderivatives for $t^{1/6}$, $\frac{1}{t}$, $\csc^2 t$, and $\frac{1}{1+t^2}$ are $\frac{t^{7/6}}{7/6}$, $\ln|t|$, $-\cot t$, and $\arctan t$ respectively, so the general antiderivative of $h(t)$ is

$$\frac{t^{7/6}}{7/6} + 5\ln|t| - 4\cot t - \arctan t + C$$

4. **(12 points)** *Answer the following questions about approximation:*

(a) **(6 points)** *Starting with an initial value of 1, use two iterations of Newton's method to approximate a zero of $f(x) = x^6 - 5x + 3$. Your answer need not be arithmetically simplified.*

Let us start by observing that $f'(x) = 6x^5 - 5$. Using Newton's method once:

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1 - \frac{1^6 - 5 \cdot 1 + 3}{6 \cdot 1^5 - 5} = 1 - \frac{-1}{1} = 2$$

And using it again:

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 2 - \frac{2^6 - 5 \cdot 2 + 3}{6 \cdot 2^5 - 5} = 2 - \frac{57}{187} = \frac{317}{187}$$

This isn't terribly close to the correct result of approximately 1.2014053, but a few more iterations would get it closer.

(b) **(6 points)** *Choose $x_1 = 4$ to be an initial approximation of $\sqrt{17}$. Use one step of Newton's method on an appropriately chosen polynomial function to develop x_2, a better rational approximation of $\sqrt{17}$; also give an arithmetic expression (which need not be simplified) for the better approximation x_3 arising from a second step of Newton's method.*

We want an easy-to-evaluate polynomial of which $\sqrt{17}$ is a zero, in order for Newton's method to help us approximate it; the obvious choice is $f(x) = x^2 - 17$. Note that $f'(x) = 2x$. Then,

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 4 - \frac{4^2 - 17}{2 \cdot 4} = 4 - \frac{-1}{8} = 4 + \frac{1}{8} = \frac{33}{8}$$

And we follow up with the further improvement

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = \frac{33}{8} - \frac{\left(\frac{33}{8}\right)^2 - 17}{2 \cdot \frac{33}{8}}$$

This last expression is actually $\frac{2177}{528}$, which is within 0.0000005 of the correct value of $\sqrt{17}$.

5. **(22 points)** *Evaluate the following limits; if they cannot be evaluated, show why not.*

(a) $\lim\limits_{x \to 0} \frac{6x}{\arctan x}$.

Note that $6 \cdot 0 = 0$ and $\arctan 0 = 0$, so this is a $\frac{0}{0}$ indeterminate form. Using L'Hôpital's rule:

$$\lim_{x \to 0} \frac{6x}{\arctan x} = \lim_{x \to 0} \frac{6}{\frac{1}{1+x^2}} = \frac{6}{\frac{1}{1+0^2}} = 6$$

(b) $\lim\limits_{\theta \to 0} \frac{\theta + \sin \theta}{\theta + \cos \theta}$.

This one can be solved by direct evaluation: $\lim\limits_{\theta \to 0} \frac{\theta + \sin \theta}{\theta + \cos \theta} = \frac{0 + \sin 0}{0 + \cos 0} = \frac{0}{1} = 0$.

(c) $\lim\limits_{u \to +\infty} \frac{e^{u/10}}{u^3}$.

As u gets very large, $e^{u/10}$ and u^3 both get very large, so this is an $\frac{\infty}{\infty}$ indeterminate form. Using L'Hôpital's rule:

$$\lim_{u \to +\infty} \frac{e^{u/10}}{u^3} = \lim_{u \to +\infty} \frac{\frac{1}{10} e^{u/10}}{3u^2}$$

which is still an $\frac{\infty}{\infty}$ indeterminate form, so we apply L'Hôpital's rule again:

$$\lim_{u \to +\infty} \frac{\frac{1}{10} e^{u/10}}{3u^2} = \lim_{u \to +\infty} \frac{\frac{1}{100} e^{u/10}}{6u}$$

And it's still an $\frac{\infty}{\infty}$ form, so perhaps the thrid time's the charm:

$$\lim_{u \to +\infty} \frac{\frac{1}{100} e^{u/10}}{6u} = \lim_{u \to +\infty} \frac{\frac{1}{1000} e^{u/10}}{6}$$

which grows without bound in the numerator with a fixed denominator, so this limit does not exist, since it grows without bound.

(d) $\lim\limits_{x \to 0} \frac{\sin x - x}{x^2 e^x}$.

Note that $\sin 0 - 0 = 0$ and $0^2 e^0 = 0$, so this is a $\frac{0}{0}$ indeterminate form. Using L'Hôpital's rule, and applying the product rule to the denominator:

$$\lim_{x \to 0} \frac{\sin x - x}{x^2 e^x} = \lim_{x \to 0} \frac{\cos x - 1}{2xe^x - x^2 e^x}$$

But this is still a $\frac{0}{0}$ indeterminate form, so applying L'Hôpital's rule again:

$$\lim_{x \to 0} \frac{\cos x - 1}{2xe^x - x^2 e^x} = \lim_{x \to 0} \frac{-\sin x}{2e^x + 4xe^x - x^2 e^x}$$

and now this is directly evaluatable to give $\frac{0}{2} = 0$.

(e) $\lim_{x\to 1} \frac{\ln x}{x^2-2x+1}$.

Note that $\ln 1 = 0$ and $1^2 - 2 \cdot 1 + 1 = 0$, so this is a $\frac{0}{0}$ indeterminate form. Using L'Hôpital's rule:

$$\lim_{x\to 1} \frac{\ln x}{x^2-2x+1} = \lim_{x\to 1} \frac{\frac{1}{x}}{2x-2} = \lim_{x\to 1} \frac{1}{x(2x-2)}$$

Since the numerator is nonzero and the denominator is zero, this limit does not exist.