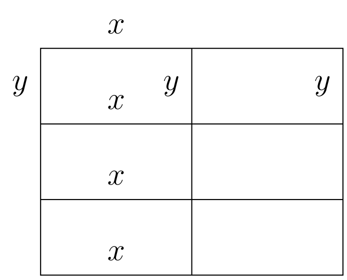
1. **(12 points)** *The keratoid cusp is a curve satisfying the equation* $y^2 = x^2y + x^5$.

   (a) **(9 points)** *Find a formula for* $\frac{dy}{dx}$ *on this curve.*
   We differentiate both sides with respect to x, and use the product and chain rule as appropriate:

$$\begin{aligned}
\frac{d}{dx}\left(y^2\right) &= \frac{d}{dx}\left(x^2y + x^5\right) \\
\frac{dy}{dx}\frac{d}{dy}y^2 &= 2xy + x^2\frac{dy}{dx} + 5x^4 \\
2y\frac{dy}{dx} &= 2xy + x^2\frac{dy}{dx} + 5x^4 \\
(2y - x^2)\frac{dy}{dx} &= 2xy + 5x^4 \\
\frac{dy}{dx} &= \frac{2xy + 5x^4}{2y - x^2}
\end{aligned}$$

   (b) **(3 points)** *Find the equation of the tangent line to the keratoid cusp at the point* $(2, -4)$.
   Using the result above at this point, we find that at $(2, -4)$, the slope of the tangent line is $\frac{dy}{dx} = \frac{2\cdot2(-4)+5\cdot2^4}{2(-4)-2^2} = \frac{64}{-12} = \frac{-16}{3}$.
   The equation of the tangent line is thus, in point-slope form, $(y + 4) = \frac{-16}{3}(x - 2)$.

2. **(12 points)** *You are constructing a subdivided rectangular pasture by fencing around the entire pasture, and then partitioning it with one fence running parallel to one pair of edges and two fences parallel to the other pair of edges, dividing the pasture into 6 sections. If the entire pasture is to have an area of 1200 square feet, what is the minimum amount of fencing you can use to build this pasture?*



The above drawing is a representation of the scenario described; we assign the two dimensions of the field the labels of x and y; note that each of the seven fences then have lengths of x or y.

Our goal is to minimize the total quantity of fencing used, subject to the condition that the area of the field is 1200 square feet. Since the total length of fencing can be seen to be $4x + 3y$, and the area of the field is xy, these correspond to the problem of minimizing $4x + 3y$ subject to the constraint that $xy = 1200$. We may re-express this constraint as $y = \frac{1200}{x}$, so that the expression we seek to minimize is, in a single variable, $f(x) = 4x + 3 \cdot \frac{1200}{x} = 4x + \frac{3600}{x}$. The interval of acceptable values on x is $[0, \infty)$, since our field cannot have negative width but has no bound on its upper length.

Since $f'(x) = 4 - \frac{3600}{x^2}$, we see that $f'(x)$ will have 3 critical points: one when $x = 0$, where $f'(x)$ is undefined, and two at the solutions to $4 - \frac{3600}{x^2} = 0$, which occur at $x = \pm\sqrt{900} = \pm 30$. Our options for maximizing choices of x are thus the 3 critical points 0, -30, and 30, together with the interval endpoints 0 and the limiting behavior as $x \to \infty$. -30 is outside the interval and may be rejected out of hand. Evaluating at the other three points, we see that $f(x)$ does not exist at $x = 0$, but that $\lim_{x \to 0^+} f(x) = +\infty$; $\lim_{x \to +\infty} f(x) = +\infty$, and $f(30) = 4 \cdot 30 + \frac{3600}{30}$. This last value need not be completely calculated; it is finite, and thus a better minimum than the other two prospects. Thus, our area-minimizing choice of x is 30, which has an associated value of $y = \frac{1200}{30} = 40$, so our optimal dimensions are 30×40.

3. **(12 points)** *G-23 paxilon hydrochlorate is eliminated from the bloodstream at a rate of 12% per hour. Miranda has just taken a 40mg intravenous dose.*

   (a) **(4 points)** *Construct a function modeling the quantity of the drug in her body after t hours.*

   Since the elimination rate is 12%, the quantity of the drug is $f(t) = 40e^{-0.12t}$.

   (b) **(4 points)** *How quickly is the drug being eliminated after 2 hours?*

   The rate of eliminaiton is denoted by $f'(t)$; here, $f'(t) = 40 \cdot (-0.12)e^{-0.12t}$, and specifically $f'(2) = 40(-0.12)e^{-0.24}$, which is approximately -3.77, i.e. an elimination rate of 3.77 mg/hour.

   (c) **(4 points)** *How many hours does it take for the quantity of the drug to drop below 10mg?*

   We solve for the value of t such that $f(t) = 10$:

$$\begin{aligned} 10 &= 40e^{-0.12t} \\ \frac{1}{4} &= e^{-0.12t} \\ \ln\frac{1}{4} &= -0.12t \\ \frac{\ln\frac{1}{4}}{-0.12} &= t \end{aligned}$$

   so there will only be 10mg remaining after $\frac{\ln\frac{1}{4}}{-0.12} \approx 11.6$ hours.

4. **(12 points)** *Let $h(x) = (x^2+1)e^x$.*

   (a) **(5 points)** *Where is $h(x)$ increasing? Where is it decreasing?*

   We calculate $h'(x) = (x^2+1)e^x + 2xe^x = (x^2+2x+1)e^x$; since $h'(x) = (x+1)^2e^x$, it is evident that $h'(x) = 0$ at $x = -1$, but is positive everywhere else, so $h(x)$ is increasing where $h'(x)$ is positive, which is to say when $x \neq -1$.

   (b) **(7 points)** *Determine $h(x)$'s concavity and identify points of inflection.*

   We calculate $h''(x) = (x^2+2x+1)e^x + (2x+2)e^x = (x^2+4x+3)e^x$; since $h''(x) = (x+1)(x+3)e^x$, it is evident that $h'(x) = 0$ at $x = -1$ and $x = -3$. Now we can probe in the three regions: when $x < -3$, when $-3 < x < -1$, and when $x > -1$, with the three following respective calculations: $h''(-4) = ((-4)^2 + 4(-4) + 3)e^{-4} = 3e^{-4}$, $h''(-2) = ((-2)^2 + 4(-2) + 3)e^{-2} = -e^{-2}$, and $h''(0) = (0^2 + 4 \cdot 0 + 3)e^0 = 3$, so the diagram of the sign of $h''(x)$ at various values is given by:

$$\begin{array}{ccccc} +++++++ & 0 & ----------- & 0 & ++++ \\ & -3 & & -1 & \end{array}$$

so $h(x)$ is concave up where $h''(x)$ is positive, which is to say when $x < -3$ or $x > -1$. $h(x)$ is concave down where $h''(x)$ is negative, which is to say when $-3 < x < -1$. The points of inflection are places where the concavity changes, which is at $x = -3$ and $x = -1$.

5. **(8 points)** *Answer the following questions about the function* $g(x) = \sqrt{25 - x^2}$.

   (a) **(4 points)** *What is the domain of* $g(x)$*?*

   $g(x)$ is defined as long as the argument of the square root is non-negative, so wherever $25 - x^2 \geq 0$, whihc is when $x^2 \leq 25$, which is in the interval $[-5, 5]$.

   (b) **(4 points)** *Where does the derivative of* $g(x)$ *exist?*

   Using the chain rule, $g'(x) = \frac{-2x}{2\sqrt{25-x^2}}$. The argument of the square root must exist, limiting our options to the domain found in part (a), but in addition it must be nonzero, so we must additionally exclude values such that $25 - x^2 = 0$, which are $x = \pm 5$. Thus, our interval of differentiability is $(-5, 5)$.

6. **(12 points)** *Consider the function* $g(x) = \frac{e^x}{x+2}$.

   (a) **(5 points)** *Identify the zeroes, vertical asymptotes, and long-term behavior on both sides of this function. Clearly label which is which.*

   Since $e^x > 0$ for all x, this function has no zeroes, and the vertical asymptote occurs when the denominator is zero, at $x = -2$. In the long term, this function's dominant terms in the numerator and denominator are e^x and x respectively, so $\lim_{x \to +\infty} g(x) = +\infty$ and $\lim_{x \to -\infty} g(x) = 0$.

   (b) **(5 points)** *Identify the critical points of this function, and indicate whether each is a local maximum, local minimum, or neither.*

   Using the quotient rule,

$$\begin{aligned} g'(x) &= \frac{(x+2)\left(\frac{d}{dx}e^x\right) - e^x\left(\frac{d}{dx}(x+2)\right)}{(x+2)^2} \\ &= \frac{(x+2)e^x - e^x}{(x+2)^2} \\ &= \frac{(x+1)e^x}{(x+2)^2} \end{aligned}$$

   While the zero of the denominator might be considered a critical point, it is not a local extremum, since the function $g(x)$ is in fact discontinuous there. Thus, the only extremum we need worry about is the zero of $(x+1)e^x$. This occurs at $x = -1$. Noting that $x + 1$ is negative if $x < -1$ and positive if $x > -1$, this extremum is a local minimum.

   (c) **(2 points)** *Which if any of the critical points identified above are global maxima or global minima? Show work or explain.*

   $x = -1$ is actually a global minimum, since $g'(-1)$ is negative, placing it below either of the long-term behaviors.

7. **(12 points)** *Evaluate the following integrals:*

   (a) **(4 points)** $\int_0^2 \frac{\sec\sqrt{x}\tan\sqrt{x}}{\sqrt{x}}dx$.

   Since we have $\sec\sqrt{x}\tan\sqrt{x}$ in the integrand, it makes sense to use the substitution $u = \sqrt{x}$ to simplify this expression to $\sec u \tan u$. We are particularly fortunate since then $\frac{du}{dx} = \frac{1}{2\sqrt{x}}$ and thus $2du = \frac{1}{\sqrt{x}}dx$, and a division by $\sqrt{x}$ is in fact present in our integral. We may now perform the u-substitution:

$$\int_0^2 \frac{\sec\sqrt{x}\tan\sqrt{x}}{\sqrt{x}}dx = \int_0^{\sqrt{2}} \sec u \tan u(2du) = 2\sec u]_0^{\sqrt{2}} = 2\sec\sqrt{2} - 2\sec 0$$

   (b) **(4 points)** $\int_{-2}^1 2x^2 + \frac{3}{x^2+1}dx$.

   Using known antiderivatives,

$$\int_{-2}^1 2x^2 + \frac{3}{x^2+1}dx = \left[\frac{2}{3}x^3 + 3\arctan x\right]_{-2}^1 = \frac{2}{3} + 3\arctan 1 - \left(\frac{-16}{3} + 3\arctan(-2)\right)$$

   (c) **(4 points)** $\int \frac{1}{x\ln x}dx$.

   Since we have $\frac{1}{\ln x}$ in the integrand, it makes sense to use the substitution $u = \ln x$ to simplify this expression to $\frac{1}{u}$. We are particularly fortunate since then $\frac{du}{dx} = \frac{1}{x}$ and thus $du = \frac{1}{x}dx$, and a division by x is in fact present in our integral. We may now perform the u-substitution:

$$\int \frac{1}{x\ln x}dx = \int \frac{1}{u}du = \ln u + C = \ln(\ln x) + C$$

8. **(12 points)** *Determine the following limits.*

   (a) **(4 points)** *Evaluate* $\lim_{x\to 0}(e^x - 1)\csc x$ *or demonstrate that it cannot be evaluated.*

   This is a $0 \times \infty$ form which can be rephrased as the $\frac{0}{0}$ indeterminate form $\lim_{x\to 0}\frac{e^x-1}{\sin x}$. Applying L'Hôpital's rule:

$$\lim_{x\to 0}\frac{e^x - 1}{\sin x} = \lim_{x\to 0}\frac{e^x}{\cos x} = \frac{1}{1} = 1$$

   (b) **(4 points)** *Using the difference quotient, find the derivative with respect to x of $f(x) = 4x^2 - x + 5$. You may not use L'Hôpital's rule for this problem.*

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0}\frac{(4(x+h)^2 - (x+h) + 5) - (4x^2 - x + 5)}{h} \\ &= \lim_{h\to 0}\frac{(4x^2 + 8xh + 4h^2 - x - h + 5) - (4x^2 - x + 5)}{h} \\ &= \lim_{h\to 0}\frac{8xh + 4h^2 - h}{h} \\ &= \lim_{h\to 0} 8x + 4h - 1 = 8x - 1 \end{aligned}$$

(c) **(4 points)** *Evaluate* $\lim_{t\to\infty} \frac{1}{t^2 e^t}$ *or demonstrate that it cannot be evaluated.*

As t gets very large, the denominator of this expression will get very large while the numerator is a constant 1; as a result, the limit is 0.

9. **(12 points)** *A sentry at Blackgate Prison has turned a spotlight on an escapee who is currently 0.3 miles to the north and 0.4 miles to the east of the prison. She notices that the escapee is traveling eastwards at four miles per hour.*

(a) **(6 points)** *How quickly will she need to rotate the spotlight to keep it trained on the escapee?*

Let the angle of the spotlight from true north be called θ, and let the distance the fugitive is to the east of the prison be called x. Since the fugitive is a constant distance 0.3 to the north of the prison, and a distance x easy of the prison, drawing a right triangle makes it clear that $\tan\theta = \frac{x}{0.3}$.

We know that x is currently 0.4, and that $\frac{dx}{dt} = 4$¡ so we may use related-rates techniques to determine $\frac{d\theta}{dt}$, differentiating each side of the above relationship with respect to t:

$$\frac{d}{dt}\tan\theta = \frac{d}{dt}\frac{x}{0.3}\,\frac{d\theta}{dt}\frac{d}{d\theta}\tan\theta \qquad = \frac{\frac{dx}{dt}}{0.3}\frac{d\theta}{dt}\sec^2\theta = \frac{\frac{dx}{dt}}{0.3}\frac{d\theta}{dt} \qquad = \frac{\frac{dx}{dt}\cos^2\theta}{0.3}$$

and since θ is in a right triangle adjacent to a side of length 0.3 and with hypotenuse 0.5, we may specifically expand that to

$$\frac{d\theta}{dt} = \frac{\frac{dx}{dt}\cos^2\theta}{0.3} = \frac{4\left(\frac{0.3}{0.5}\right)^2}{0.3} = \frac{1.2}{0.25} = 4.8$$

in radians per hour.

(b) **(6 points)** *How quickly is the escapee's distance from the prison changing?*

$$\begin{aligned}
\frac{d}{dt}(0.3^2 + x^2) &= \frac{d}{dt}(s^2) \\
\frac{dx}{dt}\frac{d}{dx}(x^2) &= \frac{ds}{dt}\frac{d}{ds}(s^2) \\
\frac{dx}{dt}(2x) &= \frac{ds}{dt}(2s) \\
\frac{x\frac{dx}{dt}}{s} &= \frac{ds}{dt}
\end{aligned}$$

And since x and s have current values of 0.4 and 0.5 respectively, $\frac{ds}{dt} = \frac{0.4\cdot 4}{0.5} = 3.2$

10. **(12 points)** *Answer the following questions:*

(a) **(4 points)** *Find* $\int \frac{d}{dt}\frac{e^t}{t}dt$.

We know that in general $\int \frac{d}{dt}f(t)dt = f(t) + C$, so in this case, $\int \frac{d}{dt}\frac{e^t}{t}dt = \frac{e^t}{t} + C$.

(b) **(4 points)** *Find* $\frac{d}{dx}\arctan\frac{x^2-1}{x+2}$.

Let $u = \frac{x^2-1}{x+2}$; we can use the chain rule and then the quotient rule:

$$\begin{aligned}
\frac{d}{dx}\arctan\frac{x^2-1}{x+2} &= \frac{d}{dx}\arctan u \\
&= \frac{du}{dx}\frac{d}{du}\arctan u \\
&= \left(\frac{d}{dx}\frac{x^2-1}{x+2}\right)\frac{1}{1+u^2} \\
&= \left(\frac{(x+2)(2x)-(x^2-1)(1)}{(x+2)^2}\right)\frac{1}{1+\left(\frac{x^2-1}{x+2}\right)^2}
\end{aligned}$$

(c) **(4 points)** *Given* $g(s) = \frac{d}{ds}e^s\cot(s^2)$*, find* $g'(s)$*.*
Using the product rule, $g(s) = e^s\cot(s^2) - 2se^s\csc^2(s^2)$. Then, using the product rule again, with several applications of the chain rule:

$$g'(s) = e^s\cot(s^2) - 4se^s\csc^2(s^2) - 2e^s\csc^2(s^2) + 8s^2e^s\csc(s^2)\csc(s^2)\cot(s^2)$$