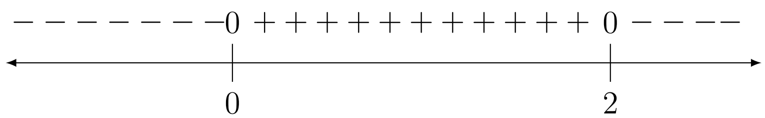
1. **(7 points)** *Let us consider the graph of the function $f(x) = 2 + 3x^2 - x^3$. Answer the following questions preparatory to sketching the curve.*

   (a) **(2 points)** *On what intervals of x-values is the function increasing? On which intervals is it decreasing? Label which is which.*

   We can easily calculate that $f'(x) = 6x - 3x^2$, which factors into $f'(x) = -3x(x-2)$, so $f'(x)$ is zero at $x = 0$ and $x = 2$. Probing between these, we see that $f'(1) = 3$, which is positive, and probing on the left and right intervals, we see that $f'(-1) = -9$ and $f'(3) = -3$, so on these intervals $f'(x)$ is negative. We thus have the sign chart:



   Thus, $f(x)$ is increasing for $0 < x < 2$, and decreasing when $x < 0$ or $x > 2$.

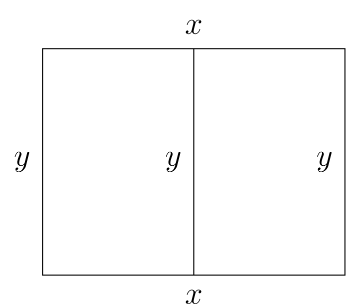
   (b) **(2 points)** *At what x-values are the local extrema, and which type of local extremum is each?*

   Since $f(x)$ is decreasing up to $x = 0$ and increasing immediately afterwards, $x = 0$ is the bottom of a "valley", i.e. a local minimum, while $f(x)$ increases to $x = 2$ and decreases therefrom, so $x = 2$ is at the top of a "hill", i.e. a local maximum.

   (c) **(3 points)** *On what intervals of x-values is the function concave up? On which is it concave down? Label which is which. Also identify the points of inflection.*

   From the above claculation of $f'(x)$, we can easily differentiate again to get $f''(x) = 6 - 6x$, which is positive if $x < 1$ and negative if $x > 1$, so $f(x)$ is concave up if $x < 1$, concave down if $x > 1$, and the transition between them, at $x = 1$, is the point of inflection.

2. **(7 points)** *We want to enclose a rectangular animal pasture with a fence all around the outside as well as one fence down the middle, parallel to a pair of sides of the pasture, so as to divide the enclosure into two rectangular sections. We have 900 feet of fencing to use. What dimensions for our pasture maximize its area?*



   The above drawing is a representation of the scenario described; we assign the two dimensions of the field the labels of x and y; note that each of the five fences then have lengths of x or y.

   Our goal is to maximize the area of the field, subject to the condition that the total quantity of fencing used is 900 feet. Since the total length of fencing can be seen to be $2x + 3y$, and the area of the field is xy, these correspond to the problem of maximizing xy subject to the constraint that $2x + 3y = 900$. We may re-express this constraint as $y = \frac{900-2x}{3} = 300 - \frac{2}{3}x$, so that the expression we seek to maximize is, in a single variable, $A(x) = x(300 - \frac{2}{3}x) = 300x - \frac{2}{3}x^2$. The interval of acceptable values on x is $[0, 450]$, since we can use no more than 900 feet even on the two fences of length x.

Since $A'(x) = 300 - \frac{4}{3}x$, we see that $A(x)$ will have one critical point at $x = 225$. Our candidates for optimization are thus the interval endpoints $x = 0$ and $x = 450$ together with this critical point 225. Since $A(0) = 0$ and $A(450) = 0$, the critical point 225 (at whech the area is actually positive) must be optimal. Thus, our area-maximizing choice of x is 225, which has an associated value of $y = 300 - \frac{2}{3} \cdot 225 = 150$, so our optimal dimensions are 225×150.

3. **(6 points)** *Determine the value of the following limits.*

   (a) $\lim\limits_{x \to +\infty} \frac{(\ln x)^2}{x}$.
   Since both $(\ln x)^2$ and x increase without bound as x increases, this is an $\frac{\infty}{\infty}$ indeterminate form. We invoke L'Hôpital's rule to get:

$$\lim_{x \to +\infty} \frac{(\ln x)^2}{x} = \lim_{x \to +\infty} \frac{\frac{d}{dx}(\ln x)^2}{\frac{d}{dx}x} = \lim_{x \to +\infty} \frac{\frac{1}{x} \cdot 2 \ln x}{1} = \lim_{x \to +\infty} \frac{2 \ln x}{x}$$

   which is still an $\frac{\infty}{\infty}$, indeterminate form, so we invoke L'Hôpital's rule again:

$$\lim_{x \to +\infty} \frac{2 \ln x}{x} = \lim_{x \to +\infty} \frac{frac2x}{1} = 0$$

   (b) $\lim\limits_{\theta \to 0} \frac{\cos \theta - 1}{\theta^2}$.
   Since $\cos 0 - 1 = 0$ and $0^2 = 0$, this is a $\frac{0}{0}$ indeterminate form, on which we may invoke L'Hôpital's rule:

$$\lim_{\theta \to 0} \frac{\cos \theta - 1}{\theta^2} = \lim_{\theta \to 0} \frac{-\sin \theta}{2\theta}$$

   which is still a $\frac{0}{0}$ indeterminate form, so again we apply L'Hôpital's rule:

$$\lim_{\theta \to 0} \frac{-\sin \theta}{2\theta} = \lim_{\theta \to 0} \frac{-\cos \theta}{2} = \frac{-1}{2}$$

   (c) $\lim\limits_{t \to 0} \frac{e^t}{t \sin t + \cos t}$. This is not an indeterminate form: $e^0 = 1$ and $0 \sin 0 + \cos 0 = 1$, so this limit may be directly evaluated to get 1.