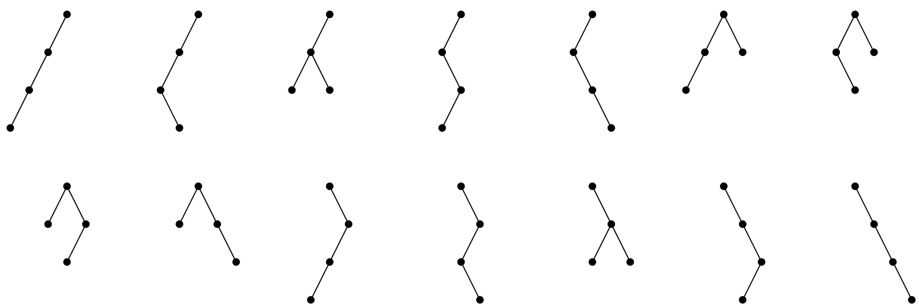
Learning to Count

1. **(10 points)** *A* rooted binary tree *is a structure consisting of nodes hierarchically arranged so that each node is be connected to either, both, or neither of a "left child" and "right child". Determine how many different rooted binary trees on 4 nodes there are by drawing all of them.*

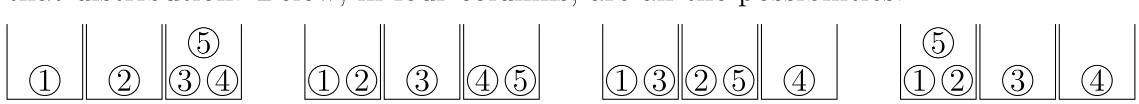
   We will make use of the following procedure: at each level, we will consider every possible set of distributions of nodes from left to right. For instance, the root node has 3 descendants; we can distribute them 3–0, 2–1, 1–2, or 0–3. Starting with the 3–0 case, we consider all possible 2–0, 1–1, and 0–2 descendant distributions for the left node, and so forth. The complete list is as follows.

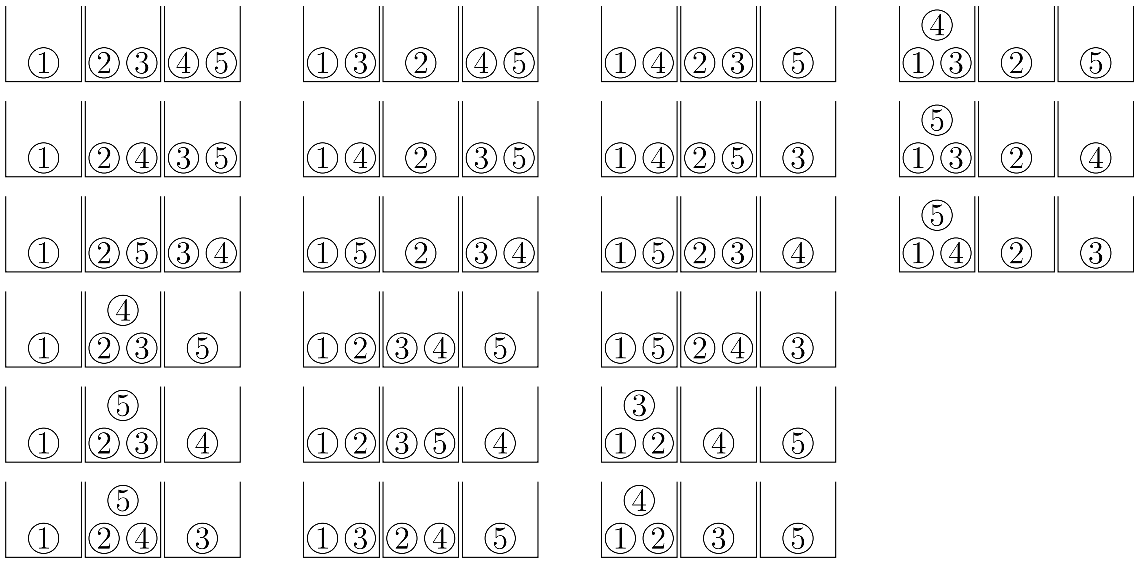


   There are 14 elements of this list. The number of rooted binary trees on n nodes is in fact given by the (fairly omnipresent) Catalan numbers (OEIS A000108), which we have encountered in passing already in describing the number of ways of nesting n pairs of parentheses. In fact, there's a straightforward bijection between parenthesis-nestings and rooted binary trees, which explains why they have the same enumeration.

2. **(10 points)** *You have 5 different balls, with the numbers from 1 to 5 written on them, and 3 indistinguishable boxes. You are putting balls into boxes so that* each box contains at least one ball*, so, for instance, you could put balls 2,4, and 5 in one box, ball 1 in a different box, and ball 3 in a third box. Determine, by listing the possibilities, how many different ways there are to distribute the balls.*

   We might arbitrarily order the boxes so that the box containing "1" is designated as the first, the box containing the lowest-numbered ball not in the first box is designated as the second, and the remaining box as the last. This gives us the luxury of drawing these indistinguishable boxes in a distinguished order with relative certainty that we will not represent the same distribution two different ways. We might methodically go through the orderings in a nubmer of different ways; here I consider classification by the number of balls in each box, since there are only a very small number of ways of cutting 5 into three positive integer parts. Here I consider in turn each of the ordered quantity-distributions $(1,1,3)$, $(1,2,2)$, $(1,3,1)$, $(2,1,2)$, $(2,2,1)$, $(3,1,1)$, and for each quantity-distribution I consider all the particular labelings of balls respecting that distribution. Below, in four columns, are all the possibilities.

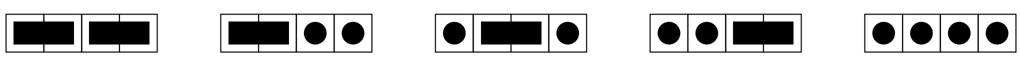




This list has a total of 25 different distributions. We will learn an easier computation for this statistic later; the relevant named value for this enumeration is that it is the Stirling number of the second kind designated $S(5,3)$.

3. **(5 points)** *You have a 1×4 checkerboard, and you want to tile all the squares with some combination of dominoes (which cover two squares) and checkers (which cover one square). List all the ways you can do this (don't forget to include the tiling using 4 checkers and no dominoes, and the tiling using 2 dominoes and no checkers!).*

   There are five such tilings, shown below. We produce this table by listing all tilings beginning with a domino first, then all beginning with a checker; subsidiary to this division, we subdivide the shorter tilings by whether their second element is a domino or checker, and so forth. Other methodical listings are possible.

   The number of domino tilings of a $1 \times n$ rectangle is, in general, given by a number in the well-loved Fibonacci sequence (OEIS A000045); we shall see why later in the course.

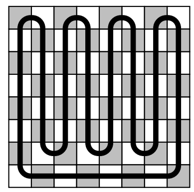
Deductive techniques

4. **(10 points)** *An odd number of people meet up, and some of them shake each other's hands. Explain why it is impossible for every person to have shaken an odd number of other people's hands.*

   Let us add up the total number of people that each person has shaken hands with, and call this sum x. Each handshake contributes 2 towards this total: if A shakes B's hand, that increments both A and B's total. Thus, since x is twice the number of handshakes, x must be even; since an even number cannot be the sum of an odd number of odd numbers, it is impossible for each person's number of handshakes to be odd.

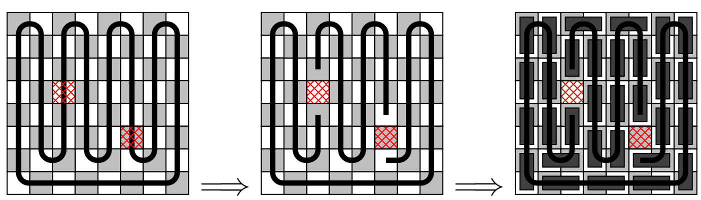
5. **(5 point bonus)** *One white and one black square are removed from an 8×8 checkerboard. Show that, regardless of which two squares are removed, the remaining 62 squares can be tiled with dominoes.*

   Showing that this is always possible is actually quite difficult! There are several approaches, but one of the most elegant is a construction based on a technique of Gomory. We start by taking a loop passing through every square of a checkerboard, as such (any loop will work; different loops might construct different tilings):

   Note that this loop necessarily traverses alternately black and white squares; cutting one square out will open the loop into just a path, and cutting out another square will divide the path into two smaller paths (possibly one of which is a trivial path of length zero. Since the squares removed are of different colors, and those squares are the termini for both paths, each path so constructed will have its two ends on squares of different colors, and thus, notably, each path will have even length. Since the length of each path is a multiple of 2, we can thus partition each path into sub-paths of length 2; each of these paths can be covered with a domino.

   Below is an illustration of how this process would construct a domino tiling for a specific removal of two squares:

Inductive techniques

6. **(15 points)** *You have a stack of discs of different sizes (which can be described by a sequence of distinct numbers) which you wish to sort in ascending order by size. An "action" consists of taking any number of discs off the top of the stack and flipping it over.*

   (a) *Explain how you could move the largest disc to the bottom of the stack in two actions.*

   Wherever in the stack the largest disc is, we can move it to the top of the stack in a single action by flipping over the section from the top down to the largest disc. For instance, in the stack 14728635, we could bring 8 to the top by flipping over the top 5 discs (since 8 is fifth from the top) to get 82741635. Once the largest disc is on the top, we can take it to the bottom with another action by flipping the entire stack over (so in our example above, we would get 53614728). Thus, in general, we can take the largest disc to the bottom with two actions.

(b) *Explain how you could move the largest disc to the bottom of the stack and then move the second-largest disc to the position immediately above the bottom in four actions.*

We start by using two actions to perform the aforementioned process to bring the largest disc to the bottom. We then use a variant of this procedure to put the second-largest disc in place: we use our third action to bring the second-largest disc to the top (by flipping the substack from the top down to the second-largest disc), and then we put this disc in its place with our fourth move by flipping over the entire stack except the bottom disc.

(c) *Using an inductive argument built on the two previous processes, explain how you could completely sort a stack of n discs in $2n$ actions.*

For an inductive argument, we start by asserting a base case: a stack of 1 disc can be sorted in 2 actions (in fact it can be sorted in 0, since a stack of one is definitionally sorted).

Now we inductively assume, as a "black box" procedure, that we have a way of sorting n discs with $2n$ actions. Using this procedure, we want to figure out how to sort a stack of $n+1$ discs with $2(n+1)$ actions. As seen above, we can use two actions to get disc $n+1$ to the bottom, and at this point we have a stack of n discs (in an unknown order) sitting on top of the disc of size $n+1$. We can now invoke the aforementioned "black box" to sort the top n discs in $2n$ actions. Thus, we can sort a stack of $n+1$ discs in $2+2n = 2(n+1)$ actions.

The question of the optimal number of actions to sort a stack of n discs is in fact an open question, and is unknown for $n > 17$! The known values appear in OEIS A058986. This is known as the "pancake problem" or the "prefix-reversal" problem, posed by "Harry Dweighter" in 1975. It is fairly easy to show that this number is between n and $2n-3$. Gates and Papadimitriou[1] improved these bounds to $\frac{17n}{16}$ below and $\frac{5(n+1)}{3}$ above. The lower bound has been improved by degrees since then, with the most recent bound being a 2009 result of Chitturi *et al.* that at least $\frac{18n}{11}$ actions are necessary.

7. **(10 points)** *Use an inductive argument to show that the sum* $2+6+12+20+30+\cdots+n(n+1)$ *is equal to* $\frac{n(n+1)(n+2)}{3}$.

We start by showing the base case to be true: when $n=1$, the former expresison is the trivial sum 2 and the latter is $\frac{1\cdot 2\cdot 3}{3}$; these are indeed equal.

Now we assume that the statement to be proven is true in a particular case, and attempt to prove the subsequent case. That is, taking it as a given that

$$2+6+12+20+30+\cdots+n(n+1) = \frac{n(n+1)(n+2)}{3}$$

[1]Christos Papadimitriou has since become one of the luminaries of discrete mathematics and theoretical computer science; William H. Gates drifted into an unremarkable career in software engineering.

we wish to show that

$$2+6+12+20+30+\cdots+(n+1)(n+2)=\frac{(n+1)(n+2)(n+3)}{3}$$

which can be done easily with arithmetic manipulations of our given statement:

$$\begin{aligned}
2+6+12+20+30+\cdots+n(n+1) &= \frac{n(n+1)(n+2)}{3}\\
2+6+12+20+30+\cdots+n(n+1)+(n+1)(n+2) &= \frac{n(n+1)(n+2)}{3}+(n+1)(n+2)\\
2+6+12+20+30+\cdots+n(n+1)+(n+1)(n+2) &= \frac{n(n+1)(n+2)+3(n+1)(n+2)}{3}\\
2+6+12+20+30+\cdots+(n+1)(n+2) &= \frac{(n+3)(n+1)(n+2)}{3}
\end{aligned}$$

Musica est exercitium arithmeticae occultum nescientis se numerare animi. [Music is an arithmetical exercise of the soul, which does not realize that it is counting.]
—Gottfried Leibniz