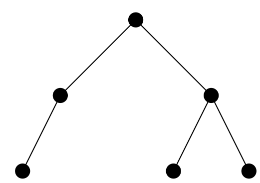
Learning to Count

In this section you need not show work, but your results should show some evidence of a methodical approach.

1. **(10 points)** A *rooted binary tree* is a structure consisting of nodes hierarchically arranged so that each node is be connected to either, both, or neither of a "left child" and "right child". For instance, this is a rooted binary tree on 6 nodes:



   Determine how many different rooted binary trees on 4 nodes there are by drawing all of them (to start off you might practice by drawing all 1 one-node trees, all 2 two-node trees, and all 5 three-node trees).

2. **(10 points)** You have 5 different balls, with the numbers from 1 to 5 written on them, and 3 indistinguishable boxes. You are putting balls into boxes so that *each box contains at least one ball*, so, for instance, you could put balls 2,4, and 5 in one box, ball 1 in a different box, and ball 3 in a third box. Determine, by listing the possibilities, how many different ways there are to distribute the balls.

3. **(5 points)** You have a 1×4 checkerboard, and you want to time all the squares with some combination of dominoes (which cover two squares) and checkers (which cover one square). List all the ways you can do this (don't forget to include the tiling using 4 checkers and no dominoes, and the tiling using 2 dominoes and no checkers!).

Deductive techniques

4. **(10 points)** An odd number of people meet up, and some of them shake each other's hands. Explain why it is impossible for every person to have shaken an odd number of other people's hands.

5. **(5 point bonus)** One white and one black square are removed from an 8×8 checkerboard. Show that, regardless of which two squares are removed, the remaining 62 squares can be tiled with dominoes.

Inductive techniques

6. **(15 points)** You have a stack of discs of different sizes (which can be described by a sequence of distinct numbers) which you wish to sort in ascending order by size. An "action" consists of taking any number of discs off the top of the stack and flipping it over; so for instance from the sequence 31452 you could flip the top 3 discs to yield 41352.

(a) Explain how you could move the largest disc to the bottom of the stack in two actions.

(b) Explain how you could move the largest disc to the bottom of the stack and then move the second-largest disc to the position immediately above the bottom in four actions.

(c) Using an inductive argument built on the two previous processes, explain how you could completely sort a stack of n discs in $2n$ actions.

7. **(10 points)** Use an inductive argument to show that the sum $2 + 6 + 12 + 20 + 30 + \cdots + n(n+1)$ is equal to $\frac{n(n+1)(n+2)}{3}$.

Musica est exercitium arithmeticae occultum nescientis se numerare animi.
—Gottfried Leibniz