Injections, Surjections, and the Pigeonhole Principle

1. **(10 points)** *Here we will come up with a sloppy bound on the number of parenthesis-nestings.*

   (a) **(5 points)** *Describe an injection from the set of possible ways to nest n pairs of parentheses to the set of bit-strings (strings consisting of the digits 0 or 1) of length $2n$.*

   The injection here will be a very simple encoding: since a nesting of n pairs of parentheses is a string of $2n$ individual parentheses, we can simply replace each open-parenthesis with a zero and a close-parenthesis with a 1, e.g. $\big(()\big)()$ will be mapped to 001101. This is clearly an injection since two different parenthesis-nestings will have different orderings of open- and close-parentheses, and will thus map to different bit-strings.

   (b) **(3 points)** *Use this injection and the known size of its range to reach a conclusion about the size of its domain, i.e. the number of parenthesis-nesting arrangements.*

   We know the domain of an injection is no larger than its range. The range here (the set of bit-strings of length $2n$) has size 2^{2n}, so we know that there are no more than 2^{2n} parenthesis-nestings.

   (c) **(2 points)** *Demonstrate that the function described above is not a bijection. How does this discovery affect the conclusion reached in part (b)?*

   This relationship is non-bijective since there are many bit-strings which do not correspond to a parenthesis-nesting; for instance, if there are an unequal number of zeroes and ones, the parentheses in a pre-image for such a string would be unbalanced. In addition, any string beginning with a one would not correspond to a parenthesis-nesting, since a parenthesis-nesting must begin with an opening parenthesis. We thus know that 2^{2n} is in fact a quite sloppy bound, and that the number of parenthesis-nestings is *less* than 2^{2n}.

   In fact, with this line of argument, you could (but need not in response to this question) pare down the range to a considerably smaller set: instead of considering all bit-strings of length $2n$, you could only consider those strings with n zeroes, n ones, a zero at the beginning, and a one at the end. This set can be seen via selection techniques to have size $\binom{2n-2}{n-1}$.

2. **(15 points)** *Let $X = \{1, 2, 3, \ldots, 100\}$, and let S be a subset of X.*

   (a) **(5 points)** *Demonstrate that there is a set S with $|S| = 50$ such that no element of S is a multiple of any other element of S.*

   A simple example is $\{51, 52, \ldots, 100\}$. Since the ratio of every pair of elements is less than 1, no element is a positive multiple of any other element.

   (b) **(10 points)** *Prove that if $|S| \geq 51$, then S must contain two numbers such that one of the numbers is a multiple of the other.*

   Let us define a function f mapping each integer to its largest odd multiple (so, e.g. 84 would be mapped to 21). Note that it will always be the case that $\frac{x}{f(x)}$ is a power of 2, since if $\frac{x}{f(x)}$ had any odd factors larger than 1, $f(x)$ wouldn't be the largest odd multiple of x. Note that f maps integers less than or equal to 100 to odd integers less than or equal to 100, of which there are exactly 50. Thus since $|S| > 50$, it is the case by the Pigeonhole Principle that $f : S \to \{1, 3, 5, \ldots, 99\}$ is not injective, so there are two elements x and y of S with the same largest odd multiple r. Since both x and y are powers of 2 times r, one of x or y is a multiple of the other.

3. **(10 points)** *Let S be an unknown set of 6 integers. Prove that it is possible, by adding and subtracting a nontrivial set of distinct elements of S, to get a multiple of 63.*

   Let us classify subsets of S according to the value of the sum of their terms modulo 63, or alternatively, according to the remainder when this sum is divided by 63 (recall that $a \equiv b \bmod 63$ when $a - b$ is a multiple of 63). Since S has 6 elements, there are 64 distinct subsets of S; since there are 63 distinct modular congruence classes (or 63 possible remainders from the division), two sets A and B have sums that are equivalent modulo 63; thus, the difference of the sums of the terms of A and the sums of the terms of B is a multiple of 63. Note that if A and B overlap this arithmetic expression may include the same number twice, once added and once subtracted; we may easily fix this expression by simply removing both occurrences without affecting the sum. Since A and B are distinct, at least one of them contains an element not appearing in the other, so the resulting arithmetic expression is nontrivial.

Basic Counting Techniques

4. **(10 points)** *Show by a casewise argument that the number of ways to put $n+2$ unlabeled balls into n labeled boxes so that there is at least one ball per box is $n + \binom{n}{2}$.*

   There are very few possible distributions, since the number of balls only slightly exceeds the number of boxes and we are required to put at least one ball in each box. There are in fact only two cases to be addressed: every box except for one could contain a single ball, while the remaining box contains three balls; or every box except for two could contain a single ball, while the remaining boxes each contain two balls. The first case comprehends n different possibilities, since any one of the n boxes could be selected as containing three balls, while the second case comprehends $\binom{n}{2}$ possibilities, since we select two boxes to be notable in containing two balls. Thus, among both cases, there are a total of $n + \binom{n}{2}$ ways to distribute the balls.

5. **(5 point bonus)** *Using an awareness of which factors are counted by the factor-counting technique seen in class and the text, determine a formula for the sum of the factors of $p^a q^b$, where p and q are prime.*

   As seen in class, a factor of $p^a q^b$ is determined by its prime factorization $p^r q^s$, where r is an integer between 0 and a inclusive, and s is an integer between 0 and b inclusive; thus we get $(a+1)(b+1)$ possible factors. These factors are fairly easy to list our methodically: they are

$$p^0q^0, p^0q^1, \ldots, p^0q^b, p^1q^0, p^1q^1, \ldots, p^1q^b, p^2q^0, p^2q^1, \ldots, p^aq^0, p^aq^1, \ldots, p^aq^b$$

And if we were to add them up, we can group them as such and factor out common terms:

$$\begin{aligned}(p^0q^0 + p^0q^1 + \cdots + p^0q^b) + (p^1q^0 + p^1q^1 + \cdots + p^1q^b) + \cdots + (p^aq^0 + p^aq^1 + \cdots + p^aq^b)\\ = p^0(q^0 + q^1 + \cdots + q^b) + p^1(q^0 + q^1 + \cdots + q^b) + \cdots + p^a(q^0 + q^1 + \cdots + q^b)\\ = (p^0 + p^1 + \cdots + p^a)(q^0 + q^1 + \cdots + q^b)\end{aligned}$$

and this is a product of finite geometric series, both of which have simple formulas, so the sum of all the factors of p^aq^b is $\frac{p^{a+1}-1}{p-1}\frac{q^{b+1}-1}{q-1}$.

This argument is easily generalized to show that if a number n has prime factorization $p_1^{e_1}p_2^{e_2}\cdots p_r^{e_r}$, then the sum of its factors is

$$\frac{(p_1^{e_1+1}-1)(p_2^{e_2+1}-1)\cdots(p_r^{e_r+1}-1)}{(p_1-1)(p_2-1)\cdots(p_r-1)}$$

Binomials and combinatorial proof

6. **(10 points+5 bonus)** *Below are two combinatorial proofs.*

   (a) **(10 points)** *Using combinatorial methods (i.e. showing both sides count the same thing), prove that*

   $$\sum_{k=0}^{n}\binom{n}{k}k = n2^{n-1}.$$

   We shall consider two different counting methods for the following structure: selection of a single distinguished number from $\{1,\ldots,n\}$ together with the selection of any subset of $\{1,\ldots,n\}$ not containing our distinguished number.

   One way we can do this is to select our distinguished x in any of n different ways, and then select a subset S of $\{1,\ldots,n\}-\{x\}$ in any of 2^{n-1} different ways. Thus, there are $n2^{n-1}$ ways to build this structure.

   Conversely, we could build this structure by, for a size k, considering a subset A of $\{1,\ldots,n\}$ such that $|A| = k$, and then selecting our distinguished x from A, and pairing x with the subset $S = A - \{x\}$. For each individual k, there are $\binom{n}{k}$ choices of A, and then since x is chosen from A, there are k choices of value for x. Thus, for a specific value of k, this procedure gives us $\binom{n}{k}k$ ways to build this structure. Since k can take on any value from zero to n, we add up all cases to get $\sum_{k=0}^{n}\binom{n}{k}k$ ways to build this structure.

   Since these two quantities are enumerating the same set, it follows that $\sum_{k=0}^{n}\binom{n}{k}k = n2^{n-1}$.

   For your entertainment — although it's not an answer to the question as asked — here's an algebraic proof of the same result using a bit of calculus and the binomial theorem:

$$\begin{aligned}
\sum_{k=0}^{n}\binom{n}{k}k &= \sum_{k=0}^{n}\binom{n}{k}kx^{k-1}\Big|_{x=1} \\
&= \sum_{k=0}^{n}\binom{n}{k}\frac{d}{dx}x^{k}\Big|_{x=1} \\
&= \left(\frac{d}{dx}\sum_{k=0}^{n}\binom{n}{k}x^{k}\right)\Big|_{x=1} \\
&= \left(\frac{d}{dx}(1+x)^{n}\right)\Big|_{x=1} \\
&= n(1+x)^{n-1}\big|_{x=1} = n2^{n-1}
\end{aligned}$$

(b) **(5 point bonus)** *Generalize your above result to argue that*

$$\sum_{k=0}^{n}\binom{n}{k}\binom{k}{\ell} = \binom{n}{\ell}2^{n-\ell}.$$

We shall consider two different counting methods for the following structure: selection of a set S of ℓ distinguished elements of $\{1,\ldots,n\}$ together with the selection of any subset T of $\{1,\ldots,n\}$ disjoint from S.

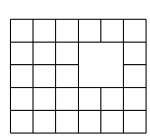
One way we can do this is to select our set S of ℓ distinguished elements in any of $\binom{n}{\ell}$ different ways, and then select the subset T of $\{1,\ldots,n\}-S$ in any of $2^{n-\ell}$ different ways. Thus, there are $\binom{n}{\ell}2^{n-\ell}$ ways to build this structure.

Conversely, we could build this structure by, for a size k, considering a subset A of $\{1,\ldots,n\}$ such that $|A|=k$, and then selecting S as an ℓ-element subset of A, and letting $T=A-S$. For each individual k, there are $\binom{n}{k}$ choices of A, and then since S is the result of choosing ℓ elements of A, from A, there are $\binom{k}{\ell}$ choices of subset S. Thus, for a specific value of k, this procedure gives us $\binom{n}{k}\binom{k}{\ell}$ ways to build this structure. Since k can take on any value from zero to n, we add up all cases to get $\sum_{k=0}^{n}\binom{n}{k}\binom{k}{\ell}$ ways to build this structure.

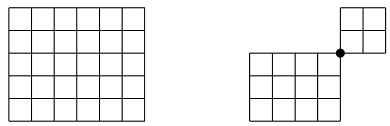
Since these two quantities are enumerating the same set, it follows that $\sum_{k=0}^{n}\binom{n}{k}\binom{k}{\ell} = \binom{n}{\ell}2^{n-\ell}$.

An algebraic proof of this result is possible but quite ugly. It is similar in spirit to the algebraic proof given for part (a) but uses the fact that $\binom{k}{\ell}x^{k-\ell} = \frac{1}{\ell!}\frac{d^{\ell}}{dx^{\ell}}x^{k}$.

7. **(5 points)** *Find the number of ways to walk from the lower left corner of the following grid to the upper right corner with sequences of "up" and "right" moves:*



This is a grid with an exclusion, so it is a matter of finding the number of gridwalks from the lower left corner to the upper left which do *not* visit the excluded point. That is, we wish to find the number of gridwalks on the left grid below, and subtract the “illegal” walks, which could be considered to be those on the right grid below:



The number of walks on the first grid is easily seen to be $\binom{11}{5}$, since a walk on this grid is a sequence of 11 moves, exactly 5 of which are upwards and 6 of which are rightwards.

A walk on the second grid can be constructed as a concatenation of walks on two individual grids: the lower left grid can be traversed in $\binom{7}{3}$ ways, and the upper left is traversable in any of $\binom{4}{2}$ ways, so this grid has $\binom{7}{3}\binom{4}{2}$ possible traversals.

Since our original problem was answerable by subtracting the number of walks on the second grid (all of which would have been illegal on our original grid) from the number of walks on the first grid, the answer will be $\binom{11}{5} - \binom{7}{3}\binom{4}{2} = 252$.

And NUH is the letter I use to spell Nutches
Who live in small caves, known as Nitches, for hutches
These Nutches have troubles, the biggest of which is
The fact there are many more Nutches than Nitches.

—Dr Seuss, *On Beyond Zebra*