Injections, Surjections, and the Pigeonhole Principle

1. **(10 points)** Here we will come up with a sloppy bound on the nubmer of parenthesis-nestings. Recall for reference that, for instance, there are 5 ways to nest 3 pairs of parentheses, namely: $\big((())\big)$, $(()())$, $(())()$, $()(())$, $()()()$.

   (a) **(5 points)** Describe an injection from the set of possible ways to nest n pairs of parentheses to the set of bit-strings (strings consisting of the digits 0 or 1) of length $2n$.

   (b) **(3 points)** Use this injection and the known size of its range to reach a conclusion about the size of its domain, i.e. the number of parenthesis-nesting arrangements.

   (c) **(2 points)** Demonstrate that the function described above is not a bijection. How does this discovery affect the conclusion reached in part (b)?

2. **(15 points)** Let $X = \{1, 2, 3, \ldots, 100\}$, and let S be a subset of X.

   (a) **(5 points)** Demonstrate that there is a set S with $|S| = 50$ such that no element of S is a multiple of any other element of S.

   (b) **(10 points)** Prove that if $|S| \geq 51$, then S must contain two numbers such that one of the numbers is a multiple of the other.

3. **(10 points)** Let S be an unknown set of 6 integers. Prove that it is possible, by adding and subtracting a nontrivial set of distinct elements of S, to get a multiple of 63. (As an example, when $S = \{2, 6, 8, 30, 51, 58\}$, the form $51 + 8 + 6 - 2$ is 63.)

Basic Counting Techniques

4. **(10 points)** Show by a casewise argument that the number of ways to put $n + 2$ unlabeled balls into n labeled boxes so that there is at least one ball per box is $n + \binom{n}{2}$.

5. **(5 point bonus)** Using an awareness of which factors are counted by the factor-counting technique seen in class and the text, determine a formula for the sum of the factors of $p^a q^b$, where p and q are prime.

Binomials and combinatorial proof

6. **(10 points+5 bonus)** Below are two combinatorial proofs.

   (a) **(10 points)** Using combinatorial methods (i.e. showing both sides count the same thing), prove that

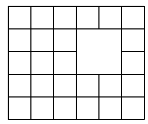
$$\sum_{k=0}^{n} \binom{n}{k} k = n 2^{n-1}$$

(b) **(5 point bonus)** Generalize your above result to argue that

$$\sum_{k=0}^{n} \binom{n}{k}\binom{k}{\ell} = \binom{n}{\ell}2^{n-\ell}$$

.

7. **(5 points)** Find the number of ways to walk from the lower left corner of the following grid to the upper right corner with sequences of "up" and "right" moves:



> And NUH is the letter I use to spell Nutches
> Who live in small caves, known as Nitches, for hutches
> These Nutches have troubles, the biggest of which is
> The fact there are many more Nutches than Nitches.
>
> —Dr Seuss, *On Beyond Zebra*