Inclusion-Exclusion

1. **(15 points+5 point bonus)** *We shall below be discussing anagrams of the word "MISSISSIPPI". Note that this word contains 4 instances of the letter "S", 4 of "I", 2 "P", and one "M".*

   (a) **(5 points)** *How many anagrams are there in total? An anagram need not be an actual English word, e.g. "ISIMPSSPISI" is a valid anagram.*

   This is a simple multinomial coefficient, $\binom{11}{4,4,2,1}$, which can, but need not be, evaluated to give 34650.

   (b) **(10 points)** *An anagram will be called* boring *if all the instances of a single letter (except M) are grouped together. For instance, "ISIMPSSPISI" is not boring, but "SIMPPSSISII" is, since both of the "P"s are together, and "SSIIIIMSPP" is since both the "I"s and "P"s are clustered. How many non-boring anagrams are there?*

   Let A_S, A_I, and A_P respectively be the set of anagrams in which the characters "S", "I", and "P" appear in a single block. $|A_S|$ can be enumerated by considering anagrams of the 8 "letter-tiles" [SSSS], [I], [I], [I], [I], [P], [P], and [M], which can be arranged in $\binom{8}{4,2,1,1}$ ways; $|A_I|$ can likewise be determined to be $\binom{8}{4,2,1,1}$ by letting [IIII] be the multi-letter tile. Since there are only two "P" tiles, $|A_P|$ will be slightly different, although the technique will be identical: $|A_P| = \binom{10}{4,4,1,1}$.

   To find the sizes of intersections, we will produce multiple "block" tiles. For instance, $A_S \cap A_I$ consists of all anagrams where all four "S" letters are adjacent and all four "I" letters are adjacent, so these anagrams can be produced by arranging the tiles [SSSS], [IIII], [P], [P], and [M], so $|A_S \cap A_I| = \binom{5}{2,1,1,1}$. Likewise, $|A_S \cap A_P| = |A_I \cap A_P| = \binom{7}{4,1,1,1}$. Lastly, the intersection $A_S \cap A_I \cap A_P$ can be characterized by anagrams of the distinct tiles [SSSS], [IIII], [PP], and [M], which can be arranged in $\binom{4}{1,1,1,1}$ ways.

   Using inclusion-exclusion, the number of boring anagrams can easily be found:

   $$\begin{aligned}|A_S \cup A_I \cup A_P| &= |A_S| + |A_I| + |A_P| - |A_S \cap A_I| - |A_S \cap A_P| - |A_I \cap A_P| + |A_S \cap A_I \cap A_P| \\ &= 2\binom{8}{4,2,1,1} + \binom{10}{4,4,1,1} - \binom{5}{2,1,1,1} - 2\binom{7}{4,1,1,1} + \binom{4}{1,1,1,1} \\ &= 1680 + 6300 - 60 - 420 + 24 = 7524\end{aligned}$$

   so the number of non-boring anagrams, using the result from the previous part, is $34650 - 7524 = 27126$.

   (c) **(5 point bonus)** *How many anagrams have* no *two instances of the same letter adjacent to each other, e.g. "SIPSISPIMIS"?*

   The number is 2016, as can be verified by a computer search. The proofs I know are really quite cumbersome with casewise analysis, but maybe you have a better one!

2. **(10 points)** *Suppose that A_1, A_2, A_3, and A_4 are 20-element subsets of $\{1, 2, \ldots, 40\}$. Show that there must be at least one pair of distinct sets A_i, A_j such that $A_i \cap A_j$ has at least 7 elements.*

   Since A_1, A_2, A_3, and A_4 are all subsets of $\{1, 2, \ldots, 40\}$, we know that $A_1 \cup A_2 \cup A_3 \cup A_4 \subseteq \{1, 2, \ldots, 40\}$. Thus, $|A_1 \cup A_2 \cup A_3 \cup A_4| \leq 40$. From inclusion-exclusion, we know that

   $$|A_1| + |A_2| + |A_3| + |A_4| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_1 \cap A_4| - |A_2 \cap A_3| - |A_2 \cap A_4| - |A_3 \cap A_4|$$

is no more than the size of $|A_1 \cup A_2 \cup A_3 \cup A_4|$ (since it is undercounting the triple and quadruple intersections). Thus, since $|A_1| = |A_2| = |A_3| = |A_4| = 20$, we know that

$$\begin{aligned} 20 + 20 + 20 + 20 - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_1 \cap A_4| - |A_2 \cap A_3| - |A_2 \cap A_4| - |A_3 \cap A_4| &\leq 40 \\ -|A_1 \cap A_2| - |A_1 \cap A_3| - |A_1 \cap A_4| - |A_2 \cap A_3| - |A_2 \cap A_4| - |A_3 \cap A_4| &\leq 40 - 80 \\ |A_1 \cap A_2| + |A_1 \cap A_3| + |A_1 \cap A_4| + |A_2 \cap A_3| + |A_2 \cap A_4| + |A_3 \cap A_4| &\geq 40 \end{aligned}$$

In order for this sum of 6 integers to be at least 40, one of them must be at least 7, since if they were all 6 or less, their sum would be 36 or less.

An observant student noticed that this same argument also works if we're only considering *three* sets. Then the final equation becomes

$$\begin{aligned} 20 + 20 + 20 - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| &\leq 40 \\ -|A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| &\leq 40 - 60 \\ |A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3| &\geq 20 \end{aligned}$$

and since these three sets have size exceeding 20, one of them must have size at least 7.

Note that there *are* sets A_1, A_2, A_3, and A_4 whose mutual intersections consist of 7 or fewer elements, so the result above cannot be improved. One example is $A_1 = \{1, 2 \ldots 20\}$, $A_2 = \{14, 15, \ldots, 33\}$, $A_3 = \{7, 8, \ldots, 13, 27, 28, \ldots, 40\}$, and $A_4 = \{1, 2, \ldots, 6, 14, 21, 22, \ldots, 26, 34, 35, \ldots, 40\}$.

3. **(10 points)** *An integer is squarefree if it is not divisible by any square number except for 1. How many squarefree positive integers are there in* $\{1, 2, 3, \ldots, 150\}$*?*

If an integer is divisible by a square number, it will be divisible specifically by the square of a prime number; thus we want to remove from consideration specifically the multiples of 2^2, 3^2, 5^2, 7^2, and 11^2. Let $X = \{1, 2, 3, \ldots, 150\}$ and let us denote the multiples of 2^2, 3^2, 5^2, 7^2, and 11^2 in X by A_4, A_9, A_{25}, A_{49}, and A_{121} respectively. We know that $|A_4| = \lfloor \frac{150}{4} \rfloor = 37$, $|A_9| = \lfloor \frac{150}{9} \rfloor = 16$, $|A_{25}| = \lfloor \frac{150}{25} \rfloor = 6$, $|A_{49}| = \lfloor \frac{150}{49} \rfloor = 3$, and $|A_{121}| = \lfloor \frac{150}{121} \rfloor = 1$. The intersections can also be characterized as multiples of certain values: $A_4 \cap A_9$ contains only multiples of 36, and $A_4 \cap A_{25}$ contains only multiples of 100. These are in fact the only two non-empty intersections: every other pair has no overlap under 150, so the total number of squarefree integers is

$$|X| - |A_4| - |A_9| - |A_{25}| - |A_{49}| - |A_{121}| + |A_4 \cap A_9| + |A_4 \cap A_{25}| = 150 - 37 - 16 - 6 - 3 - 1 + 4 + 1 = 92$$

4. **(10 points)** *Find a formula for the permutations of the numbers* $\{1, 2, 3, \ldots, n\}$ *with some number* i *appearing directly before its successor* $i + 1$*. (for example, when* $n = 4$*, the answer would be 13, specifically enumerating the permutations* 1234, 1243, 1342, 1423, 2134, 2314, 2341, 3124, 3412, 3421, 4123, 4231, 4312*)*

For each i from 1 to $n-1$, let A_i consist of all permutations of $\{1, 2, 3, \ldots, n\}$ in which i appears directly before $i+1$. We may consider such a permutation to be simply a permutation of $n - 1$ "tiles", most of this consist of a single number from 1 to n, but with i and $i+1$ on a single tile; thus $|A_i| = (n-1)!$ for all i. Similarly, if we consider the intersection $|A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}|$, then this would contain permutations of $n - k$ "tiles" formed by merging the n original numbers into blocks k times. Thus, $|A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}| = (n - k)!$ regardless of the values of $i_1, \ldots, i_k$. Thus by inclusion-exclusion:

$$|A_1 \cup A_2 \cup \cdots \cup A_{n-1}| = \sum_{k=1}^{n-1} (-1)^{k-1} \binom{n-1}{k} (n-k)!$$

Note that the example given would confirm our result; if $n = 4$ we would have $\binom{3}{1}3! - \binom{3}{2}2! + \binom{3}{3}1! = 13$.

Generating Functions

5. **(15 points)** *Produce generating functions to enumerate the following structures; explain what each term of the generating function represents. You need not algebraically expand your results.*

   (a) **(5 points)** *Four-digit numbers whose digits add up to n.*

   The choice of first digit can be anything from 1 to 9; since n is tracking the total of the digits, we place the digit in the exponent, getting $(x^1 + x^2 + x^3 + \cdots + x^9)$ as the function describing this selection.

   The second through fourth digits are between zero and 9, and thus they have selection function $(x^0 + x^1 + \cdots + x^9)$.

   Multiplying these choice functions, we have a total of

$$(x + x^2 + x^3 + \cdots + x^9)(1 + x + x^2 + \cdots + x^9) = \frac{x(1-x^9)}{1-x}\left(\frac{1-x^{10}}{1-x}\right)^3$$

   (b) **(5 points)** *The number of ways to distribute n identical balls into 7 distinct boxes so that the ith box has at least i balls in it.*

   The first ball must have at least one ball in it, and can have any number, so its selection function is $(x^1 + x^2 + x^3 + \cdots)$. Likewise, the second box has selection function $(x^2 + x^3 + x^4 + \cdots)$, and so forth up to the final box, which has selection function $(x^7 + x^8 +^9 + \cdots)$ Multiplying them all:

$$\prod_{i=1}^{7}(x^i + x^{i+1} + x^{i+2} + \cdots) = \prod_{i=1}^{7}\frac{x^i}{1-x} = \frac{x \cdot x^2 \cdot x^3 \cdot x^4 \cdot x^5 \cdot x^6 \cdot x^7}{(1-x)^7} = \frac{x^{28}}{(1-x)^7}$$

   (c) **(5 points)** *The number of ways to make change for n dollars with 5-dollar bills and 2-dollar bills.*

We can use any number of five-dollar bills (including zero), and each contributes 5 towards our total, so the selection function is $1 + x^5 + x^{10} + x^{15} + \cdots$; likewise, for two-dollar bills, our selection function would be $1 + x^2 + x^4 + x^6 + \cdots$. Thus, our generating function is

$$\frac{1}{1-x^5}\frac{1}{1-x^2}$$

6. **(5 point bonus)** *Let $a_n = 1 + 4 + 9 + \cdots + n^2$; show that the generating function $\sum_{n=0}^{\infty} a_n x^n$ is equal to $\frac{x^2+x}{(1-x)^4}$.*

   We can expand this into a double sum. A classic trick when we have a double sum is to reverse the summation order.

$$\begin{aligned}
\sum_{n=0}^{\infty} a_n x^n &= \sum_{n=0}^{\infty} \sum_{m=0}^{n} m^2 x^n \\
&= \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} m^2 x^n \\
&= \sum_{m=0}^{\infty} m^2 \sum_{n=m}^{\infty} x^n \\
&= \sum_{m=0}^{\infty} m^2 \frac{x^m}{1-x} = \frac{1}{1-x} \sum_{m=0}^{\infty} m^2 x^m
\end{aligned}$$

The sum here can be determined by clever use of derivatives of known sums, as derived below:

$$\begin{aligned}
\sum_{m=0}^{\infty} x^m &= \frac{1}{1-x} \\
\sum_{m=0}^{\infty} m x^{m-1} &= \frac{d}{dx} \frac{1}{1-x} = \frac{1}{(1-x)^2} \\
\sum_{m=0}^{\infty} m x^m = x \sum_{m=0}^{\infty} m x^{m-1} &= \frac{x}{(1-x)^2} \\
\sum_{m=0}^{\infty} m^2 x^{m-1} &= \frac{d}{dx} \frac{x}{(1-x)^2} = \frac{1-x^2}{(1-x)^4} = \frac{1+x}{(1-x)^3} \\
\sum_{m=0}^{\infty} m^2 x^m = x \sum_{m=0}^{\infty} m^2 x^{m-1} &= \frac{x+x^2}{(1-x)^3}
\end{aligned}$$

and thus the above-determined sum is

$$\sum_{n=0}^{\infty} a_n x^n = \frac{1}{1-x} \sum_{m=0}^{\infty} m^2 x^m = \frac{1}{1-x} \frac{x+x^2}{(1-x)^3} = \frac{x+x^2}{(1-x)^4}$$

שתי אבנים בונות שני בתים: שלש אבנים בונות ששה בתים: ארבע אבנים ב ונות ארבעה ועשרים בתים: חמש אבנים בונות מאה ועשרים בתים: שש אבנים בונות ש בע מאות ועשרים בתים: שבע אבנים בונות חמשת אלפים וארבעים בתים: מכאן ואילך צא וחשוב מה שאין הפה יכול לדבר ואין האוזן יכולה לשמוע

[Two stones (or letters) build two houses (or words), three stones build six houses, four stones build twenty-four houses, five stones build one hundred twenty houses, six stones build seven and twenty houses, seven stones build five thousand forty houses; thenceforth are numbers which the mouth can not speak and the ear can not hear.]

—ספר היצירה פרק ד׳ משנה ט״ז [Sefer Yetzira, Chapter 4, Verse 16]

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