

Inclusion-Exclusion

1. **(15 points+5 point bonus)** We shall below be discussing anagrams of the word “MISSISSIPPI”. Note that this word contains 4 instances of the letter “S”, 4 of “I”, 2 “P”, and one “M”.
 - (a) **(5 points)** How many anagrams are there in total? An anagram need not be an actual English word, e.g. “ISIMPSSPISI” is a valid anagram.
 - (b) **(10 points)** An anagram will be called *boring* if all the instances of a single letter (except M) are grouped together. For instance, “ISIMPSSPISI” is not boring, but “SIMPPSSISII” is, since both of the “P”s are together, and “SSIIIMSPPP” is since both the “I”s and “P”s are clustered. How many non-boring anagrams are there?
 - (c) **(5 point bonus)** How many anagrams have *no* two instances of the same letter adjacent to each other, e.g. “SIPSISPIMIS”?
2. **(10 points)** Suppose that $A_1, A_2, A_3,$ and A_4 are 20-element subsets of $\{1, 2, \dots, 40\}$. Show that there must be at least one pair of distinct sets A_i, A_j such that $A_i \cap A_j$ has at least 7 elements.
3. **(10 points)** An integer is squarefree if it is not divisible by any square number except for 1. How many squarefree positive integers are there in $\{1, 2, 3, \dots, 150\}$? Do not answer this question by brute force examination of every individual number from 1 to 150!
4. **(10 points)** Find a formula for the permutations of the numbers $\{1, 2, 3, \dots, n\}$ with some number i appearing directly before its successor $i + 1$. (for example, when $n = 4$, the answer would be 13, specifically enumerating the permutations 1234, 1243, 1342, 1423, 2134, 2314, 2341, 3124, 3412, 3421, 4123, 4231, 4312)

Generating Functions

5. **(15 points)** Produce generating functions to enumerate the following structures; explain what each term of the generating function represents. You need not algebraically expand your results.
 - (a) **(5 points)** Four-digit numbers whose digits add up to n .
 - (b) **(5 points)** The number of ways to distribute n identical balls into 7 distinct boxes so that the i th box has at least i balls in it.
 - (c) **(5 points)** The number of ways to make change for n dollars with 5-dollar bills and 2-dollar bills.
6. **(5 point bonus)** Let $a_n = 1 + 4 + 9 + \dots + n^2$; show that the generating function $\sum_{n=0}^{\infty} a_n x^n$ is equal to $\frac{x^2+x}{(1-x)^4}$.

שתי אבנים בונות שני בתים: שלש אבנים בונות ששה בתים: ארבע אבנים ב ונות ארבעה ועשרים בתים: חמש אבנים בונות מאה ועשרים בתים: שש אבנים בונות ש בע מאות ועשרים בתים: שבע אבנים בונות חמשת אלפים וארבעים בתים: מכאן ואילך צא וחשוב מה שאין הפה יכול לדבר ואין האוזן יכולה לשמוע — ספר היצירה פריץ ד' משנה ט"ז