1. **(20 points)** *Let a_n represent the number of ways to distribute n unlableled balls among 3 distinguishable boxes such that each box contains at least 2 and no more than 7 balls.*

   (a) **(5 points)** *Without explicitly calculating the ordinary generating function $\sum_{n=0}^{\infty} a_n z^n$, determine whether it is a polynomial or an infinite series. State and interpret the degree of the lowest-degree nonzero term; if it is a polynomial, then also state and interpret the degree of the highest-degree nonzero term.*

   According to the given distribution criterion, the fewest balls that can actually be distributed is 6 (2 to each box), and the most balls that can be distributed is 21 (7 to each box); thus the only nnzero terms of this sequence are a_6 through a_{21}, and so the generating function consists solely of terms of degree 6 through 21; specifically, it will be a twenty-first-degree polynomial with smallest term of degree 6.

   (b) **(5 points)** *Give a formula for the generating function $\sum_{n=0}^{\infty} a_n z^n$.*

   The generating function describing the ball-selection for each box is $z^2 + z^3 + z^4 + z^5 + z^6 + z^7 = \frac{z^2 - z^8}{1-z}$. Since there are three boxes, the boxes will all be filled when this task is completed three times, and so the generating function describing this procedure (and thus this sequence) is

   $$\left(\frac{z^2 - z^8}{1-z}\right)^3 = \frac{z^6 - 3z^{12} + 3z^{18} - z^{24}}{(1-z)^3}$$

   (c) **(10 points)** *Using the generating function from the previous part, determine specifically how many ways there are to distribute 10 balls among 3 boxes with each box containing at least 2 and no more than 7 balls.*

   We want the coefficient specifically of z^{10} in this generating function; let us expand the generating function as such:

   $$\frac{z^6 - 3z^{12} + 3z^{18} - z^{24}}{(1-z)^3} = (z^6 - 3z^{12} + 3z^{18} - z^{24}) \sum_{n=0}^{\infty} \binom{n+3-1}{3-1} z^n$$

   and note that the only term in this product which provides a z^{10} term is $z^6 \cdot \binom{4+3-1}{3-1} z^4 = \binom{6}{2} z^{10} = 15$.

   Note that this result could also be obtained fairly easily with direct enumeration (the condition that no box contains more than 7 balls ends up being a red herring, as 10 balls cannot be distributed with at least 2 per box in such a way as to violate it), or even exhaustive search.

2. **(10 points)** *Find the ordinary generating functions described below:*

   (a) **(5 points)** $\sum_{n=0}^{\infty} a_n z^n$, *where a_n is the number of ways to write n as an unordered sum of positive integers no larger than 4 (e.g. a_6 would be 9, counting the possibilities $4+2$, $4+1+1$, $3+3$, $3+2+1$, $3+1+1+1$, $2+2+2$, $2+2+1+1$, $2+1+1+1+1$, and $1+1+1+1+1+1$).*

   We can have any number of 1s, each of which contributes 1 towards the total, so the selection function for the number of 1s is $1z^0 + 1z^1 + 1z^2 + \cdots = \frac{1}{1-z}$. Likewise, we can have any number of 2s, each of which contributes 2 towards the total, so the selection function for the number of 2s is $1z^0 + 1z^2 + 1z^4 + \cdots = \frac{1}{1-z^2}$. In a similar fashion we can determine the selection function for the number of 3s and 4s to be $\frac{1}{1-z^3}$ and $\frac{1}{1-z^4}$, so the generating function for the partitioning of a number into any number of 1s, 2s, 3s, and 4s is

$$\frac{1}{(1-z)(1-z^2)(1-z^3)(1-z^4)}.$$

(b) **(5 points)** $\sum_{n=0}^{\infty} b_n z^n$, *where* b_n *is the number of ways to write* n *as an unordered sum of exactly four positive integers (e.g.* b_7 *would be 3, counting the possibilities* $4+1+1+1$, $3+2+1+1$, $2+2+2+1$*).*

Since the transpose is a bijective operation, we could count the partitions satisfying this criterion by counting their transposes instead. Since the transpose of a partition into k parts (i.e. a Ferrers diagram with k rows) is a partition whose largest part is k (i.e. a Ferrers diagram with first row of length k), the enumeration b_n can be rewritten as "the number of ways to write n as an unordered sum of positive integers no greater than 4, at least one of which is exactly 4". This is a slight variant on the partition described in the previous part of this question; the selection functions for the number of 1s, 2s, and 3s are the same as before, but now since we must have at least one 4, the selection function for including 4s in the sum is $1z^4 + 1z^8 + 1z^{12} + \cdots = \frac{z^4}{1-z^4}$, so the generating function for this enumeration is

$$\frac{z^4}{(1-z)(1-z^2)(1-z^3)(1-z^4)}.$$

3. **(10 points+5 point bonus)** *Prove the following equivalences:*

(a) **(10 points)** *There are the same number of partitions of* n *into even summands as there are partitions of* n *into summands each of which appears an even number of times.*

We could prove this via explicit bijection, or by equivalence of generating functions.

For an explicit bijection, we could note that given the partition into even summands $n = 2i_1 + 2i_2 + 2i_3 + \cdots + 2i_r$, we could easily associate it with the partiton $n = i_1 + i_1 + i_2 + i_2 + i_3 + i_3 + \cdots + i_r + i_r$, in which each summand appears an even number of times (for any j, if $2j$ appeared s times in the original partition, then j appears $2s$ times in this new paritition). This is a bijective correspondence since it is reversible: if every summand appears an even number of times in a partition, we can pair each summand i with another instance of i and "merge" them into a single summand $2i$.

Alternatively, we could work out the generating functions, and note that they are identical. The generating function for the number of partitions with only even summands can be taken as the product of the selection functions for 2s, 4s, 6s, 8s, etc. to be

$$\frac{1}{(1-z^2)(1-z^4)(1-z^6)\cdots} = \prod_{i=1}^{\infty} \frac{1}{1-z^{2i}}$$

However, the generating function for the number of partitions with an even number of each summand could be computed as the product of the selection functions for choosing an even number of each possible summand. When selecting 1s, for instance, the selection function would be $1z^0 + 0z^1 + 1z^2 + 0z^3 + 1z^4 + 0z^5 + \cdots = 1 + z^2 + z^4 + \cdots = \frac{1}{1-z^2}$. Likewise, the selection function for selecting an even number of 2s would be $\frac{1}{1-z^4}$, and so forth, so that this generating function would in fact be identical to the previously-computed generating function for the partitions with even summands.

(b) **(5 point bonus)** *There are the same number of partitions of* n *into summands none of which are divisible by 3 as there are partitions of* n *into summands each of which appears no more than twice.*

There is a quite pleasing generating-function approach. The generating function for the number of partitions of n into summands none of which are divisible by 3 can be written as a product of the individual generating functions selecting the number of ks: $z^0 + z^k + z^{2k} + z^{3k} + \cdots = \frac{1}{1-z^k}$ for each k not divisible by 3, so the generating function for the number of partitions of n into summands none of which are divisible by 3 can be written as

$$\begin{aligned}\frac{1}{(1-z)(1-z^2)(1-z^4)\cdots} &= \frac{(1-z^3) \qquad\qquad (1-z^6) \qquad\qquad \cdots}{(1-z)(1-z^2)(1-z^3)(1-z^4)(1-z^5)(1-z^6)(1-z^7)\cdots} \\ &= \frac{1-z^3}{1-z}\cdot\frac{1-z^6}{1-z^2}\cdot\frac{1-z^9}{1-z^3}\cdots \\ &= (1+z+z^2)(1+z^2+z^4)(1+z^3+z^6)\cdots\end{aligned}$$

which is exactly the generating function describing the number of partitions of n such that each summand appears no more than twice, since it is the product of selection functions permitting each summand to be present zero, one, or two times.

There may also be a combinatorial way of looknig at this problem, involving the crafting of an explicit bijection; it would be interesting to try to do so!

4. **(10 points)** *Find the exponential generating function $\sum_{n=0}^{\infty} a_n \frac{z^n}{n!}$ for the number of ways to distribute n different objects to six jugglers, if each juggler receives between three and five objects.*

The exponential generating function for the nubmer of ways to distribute objects to a single juggler is $1\frac{z^3}{3!} + 1\frac{z^4}{4!} + 1\frac{z^5}{5!}$. The generating function describing the number of ways to distribute objects to all six is then the product of their six individual functions, which is

$$\left(\frac{z^3}{3!} + \frac{z^4}{4!} + \frac{z^5}{5!}\right)^6.$$

5. **(10 points)** *Let the sequence a_n be defined by the recurrence relation $a_0 = 2$, $a_1 = 3$, and $a_n = 2a_{n-1} + 15a_{n-2}$ for $n > 1$. Answer the following questions. You may answer the second part first, if you prefer to use it to answer the first question.*

   (a) **(5 points)** *Find a closed form for a_n.*

   Since this is a homogeneous recurrence relation, any linear combination of solutions to the recurrence without initial conditions is a solution to the recurrence. Let us consider a prospective solution $a_n = r^n$; then the recurrence $a_n = 2a_{n-1} + 15a_{n-2}$ can be rewritten as $r^n = 2r^{n-1} + 15r^{n-2}$, or $r^2 - 2r - 15 = 0$, which is true when $r = 5$ or $r = -3$, so both $a_n = 5^n$ and $a_n = (-3)^n$ are possible solutions to the recurrence alone, as is $a_n = k \cdot 5^n + \ell(-3)^n$.

   To satisfy the initial conditions, it must be the case that $2 = a_0 = k + \ell$ and that $3 = a_1 = 5k - 3\ell$. Solving this system of equations gives $k = \frac{9}{8}$ and $\ell = \frac{7}{8}$, so the closed form for a_n is $\frac{9 \cdot 5^n + 7(-3)^n}{8}$.

   This problem can also be done by calculating the generating function first; the derivation for this method appears at the end of part (b).

   (b) **(5 points)** *Find a closed form for the ordinary generating function $\sum_{n=0}^{\infty} a_n z^n$.*

   One could build this from the solution to part (a), or build it from the given recurrence and values. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Then, by the recurrence:

$$\begin{aligned}
f(z) &= a_0 + a_1 z + \sum_{n=2}^{\infty} a_n z^n \\
&= 2 + 3z + \sum_{n=2}^{\infty} (2a_{n-1} + 15a_{n-2}) z^n \\
&= 2 + 3z + 2z \sum_{n=2}^{\infty} a_{n-1} z^{n-1} + 15z^2 \sum_{n=2}^{\infty} 15a_{n-2} z^{n-2} \\
&= 2 + 3z + 2z(f(z) - 2) + 15z^2 f(z) \\
&= 2 - z + 2zf(z) + 15z^2 f(z) \\
(1 - 2z - 15z^2) f(z) &= 2 - z \\
f(z) &= \frac{2 - z}{1 - 2z - 15z^2}
\end{aligned}$$

Note that if one had done this part first, then part (a) could be completed by finding coefficients in the power-series expansion of $f(z)$. We would achieve this by the method of partial fractions:

$$\frac{2 - z}{1 - 2z - 15z^2} = \frac{2 - z}{(1 - 5z)(1 + 3z)} = \frac{A}{1 - 5z} + \frac{B}{1 + 3z}$$

The associated polynomial equivalence $(1 + 3z)A + (1 - 5z)B = 2 - z$ has solution $A = \frac{9}{8}, B = \frac{7}{8}$, so

$$f(z) = \frac{\frac{9}{8}}{1 - 5z} + \frac{\frac{7}{8}}{1 + 3z} = \frac{9}{8} \sum_{n=0}^{\infty} 5^n z^n + \frac{7}{8} \sum_{n=0}^{\infty} (-3)^n z^n = \sum_{n=0}^{\infty} \frac{9 \cdot 5^n + 7(-3)^n}{8} z^n$$

and since a_n is equal to the coefficient of z^n in the series representation of $f(z)$, it follows that $a_n = \frac{9 \cdot 5^n + 7(-3)^n}{8}$ (as was alredy seen by a different method above).

6. **(10 point bonus)** *For $F_0 = 1$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n > 1$, prove the following identities combinatorially (hint: use domino-and-checker tilings of 1-row checkerboards):*

   (a) **(5 point bonus)** $F_{2n} = F_n^2 + F_{n-1}^2$.

   We know that F_{2n} represents the number of ways to tile a $1 \times 2n$ checkerboard with dominoes and checkers. We may enumerate the same set by a different means via a casewise analysis on the status of the edge between the nth and $(n + 1)$th square:

   Case I: there is not a single domino covering both the nth and $(n+1)$th square. Then we might conceive of this checkerboard covering as simply two $1 \times n$ checkerboard coverings placed end-to-end (imagine that we simply cut the checkerboard at the edge between the nth and $(n+1)$th squares). There are F_n ways that the first $1 \times n$ checkerboard could be covered, and F_n ways that the second checkerboard could be covered, so these pairs of checkerboards could be covered in a total of $F_n \times F_n = F_n^2$ different ways; thus there are F_n^2 coverings of the $1 \times 2n$ checkerboard described by this case.

   Case II: there is a single domino covering both the nth and $(n + 1)$th square. Then we might conceive of this checkerboard covering as simply a $1 \times (n - 1)$ checkerboard covering, followed by a single domino, followed by another $1 \times (n-1)$ checkerboard covering placed end-to-end (imagine that we simply cut the checkerboard at the edges between the $n-1$th and nth squares and between the nth and $(n+1)$th squares). We proceed as seen in case I and find that each sub-checkerboard can be covered in any of F_{n-1} ways, so there are F_{n-1}^2 possible coverings described in this case.

Since every one of the checkerboards enumerated by F_{2n} falls into exactly one of these two cases, it thus follows that $F_{2n} = F_n^2 + F_{n-1}^2$.

A variant of this argument (where we break at the kth domino instead of at the nth) can be used to show that, in general, $F_{k+\ell} = F_k F_\ell + F_{k-1} F_{\ell-1}$; the argument above is the special case where $k = \ell$.

(b) **(5 point bonus)** $F_n = \binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} + \cdots + \binom{\lceil \frac{n}{2} \rceil}{\lfloor \frac{n}{2} \rfloor}$.

We know that F_n represents the number of ways to tile a $1 \times n$ checkerboard with dominoes and checkers. We may divide this into cases based on how many dominoes are present. Note that if we have k dominoes, then we they cover $2k$ spaces, leaving $n-2k$ spaces for checkers. Thus, we are placing in order $n-k$ objects, consisting of k dominoes and $n-2k$ checkers. Thus, there are $\binom{n-k}{k}$ ways to place these objects, sicne we select k of the $n-k$ positions in the ordering to contain dominoes. Since k can take on any value from 0 to $\lfloor \frac{n}{2} \rfloor$, we thus know that

$$F_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k}$$

On two occasions I have been asked — "Pray, Mr. Babbage, if you put into the machine wrong figures, will the right answers come out?" In one case a member of the Upper, and in the other a member of the Lower House put this question. I am not able rightly to apprehend the kind of confusion of ideas that could provoke such a question. —Charles Babbage