1. **(20 points)** Let a_n represent the number of ways to distribute n unlableled balls among 3 distinguishable boxes such that each box contains at least 2 and no more than 7 balls.
   (a) **(5 points)** Without explicitly calculating the ordinary generating function $\sum_{n=0}^{\infty} a_n z^n$, determine whether it is a polynomial or an infinite series. State and interpret the degree of the lowest-degree nonzero term; if it is a polynomial, then also state and interpret the degree of the highest-degree nonzero term.
   (b) **(5 points)** Give a formula for the generating function $\sum_{n=0}^{\infty} a_n z^n$.
   (c) **(10 points)** Using the generating function from the previous part, determine specifically how many ways there are to distribute 10 balls among 3 boxes with each box containing at least 2 and no more than 7 balls.
2. **(10 points)** Find the ordinary generating functions described below:
   (a) **(5 points)** $\sum_{n=0}^{\infty} a_n z^n$, where a_n is the number of ways to write n as an unordered sum of positive integers no larger than 4 (e.g. a_6 would be 9, counting the possibilities $4+2$, $4+1+1$, $3+3$, $3+2+1$, $3+1+1+1$, $2+2+2$, $2+2+1+1$, $2+1+1+1+1$, and $1+1+1+1+1+1$).
   (b) **(5 points)** $\sum_{n=0}^{\infty} b_n z^n$, where b_n is the number of ways to write n as an unordered sum of exactly four positive integers (e.g. b_7 would be 3, counting the possibilities $4+1+1+1$, $3+2+1+1$, $2+2+2+1$).
3. **(10 points+5 point bonus)** Prove the following equivalences:
   (a) **(10 points)** There are the same number of partitions of n into even summands as there are partitions of n into summands each of which appears an even number of times.
   (b) **(5 point bonus)** There are the same number of partitions of n into summands none of which are divisible by 3 as there are partitions of n into summands each of which appears no more than twice.
4. **(10 points)** Find the exponential generating function $\sum_{n=0}^{\infty} a_n \frac{z^n}{n!}$ for the number of ways to distribute n different objects to six jugglers, if each juggler receives between three and five objects.
5. **(10 points)** Let the sequence a_n be defined by the recurrence relation $a_0 = 2$, $a_1 = 3$, and $a_n = 2a_{n-1} + 15a_{n-2}$ for $n > 1$. Answer the following questions. You may answer the second part first, if you prefer to use it to answer the first question.
   (a) **(5 points)** Find a closed form for a_n.
   (b) **(5 points)** Find a closed form for the ordinary generating function $\sum_{n=0}^{\infty} a_n z^n$.
6. **(10 point bonus)** For $F_0 = 1$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n > 1$, prove the following identities combinatorially (hint: use domino-and-checker tilings of 1-row checkerboards):
   (a) **(5 point bonus)** $F_{2n} = F_n^2 + F_{n-1}^2$.
   (b) **(5 point bonus)** $F_n = \binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} + \cdots + + \binom{\lceil \frac{n}{2} \rceil}{\lfloor \frac{n}{2} \rfloor}$.

On two occasions I have been asked — "Pray, Mr. Babbage, if you put into the machine wrong figures, will the right answers come out?" In one case a member of the Upper, and in the other a member of the Lower House put this question. I am not able rightly to apprehend the kind of confusion of ideas that could provoke such a question. —Charles Babbage