1. **(15 points)** A string of numbers is called "pleasant" if it consists of some (possibly zero) number of "0"s and "1"s (in any order) followed by some (possibly zero) number of "22"s, "33"s, "44"s, "5"s, and "6"s (in any order). For instance, "0110122565533" is a pleasant string of length 13. Let a_n represent the number of pleasant strings of length n. Note that $a_0 = 1$ (since the null string is pleasant) and $a_1 = 4$ (since "0", "1", "5", and "6" are pleasant strings).
   (a) **(5 points)** Explain why a_n is subject to the recurrence $a_n = 2a_{n-1} + 3a_{n-2} + 2^n$ for $n \geq 2$.
   (b) **(5 points)** Give a closed-form formula for a_n.
   (c) **(5 points)** Find a closed form for the ordinary generating function $\sum_{n=0}^{\infty} a_n z^n$. If you wish, you may do this part prior to (b) and use it to answer part (b).
2. **(10 points + 5 point bonus)** I have chosen an integer between 1 and n, and when you guess a number, I will tell you whether my number is higher, lower, or equal.
   (a) **(10 points)** Specifically describe, in step-by-step detail, a procedure to figure out what my number is with $\lceil \log_2 n \rceil$ or fewer guesses.
   (b) **(5 point bonus)** Explain why, in general, it would be impossible to guess my number in fewer than $\lceil \log_2(n+1) \rceil - 1$ guesses.
3. **(15 points)** The binary form of a number is its value written out in base 2. For instance, 37 has binary form 100101, because it is $1 \cdot 2^5 + 0 \cdot 2^4 + 0 \cdot 2^3 + 1 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0$. In the following question, outputting a 0 or 1 is a single operation.
   (a) **(5 points)** Write, in detail, a procedure which, given a number n, will output its binary form. If you find it easier, you may choose to write the number backwards: i.e. for 37, you could also write a procedure which outputs 101001.
   (b) **(10 points)** How many operations will your procedure use to write out a number n? Explain your reasoning.
4. **(10 points)** Explain why every simple graph must contain two vertices of the same degree.
5. **(10 points)** Let the vertices of a graph G have degree between δ and Δ. Show that $\delta \leq \frac{2\|G\|}{|G|} \leq \Delta$.
6. **(5 point bonus)** The complement G^c of a simple graph G is a graph on the same vertex-set as G, but such that every pair of adjacent vertices is non-adjacent in G^c, and vice versa. A graph is called *self-complement* if it is isomorphic to its own complement; for instance, the cycle C_5 is self-complement. Explain why, if G is self-complement, it must be the case that $|G|$ is either a multiple of 4 or one more than an multiple of 4.

> Die ganzen Zahlen hat der liebe Gott gemacht: alles andere ist Menschenwerk.
> —attributed to Leopold Kronecker