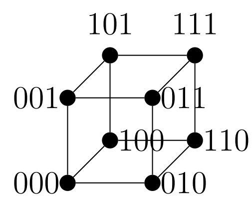
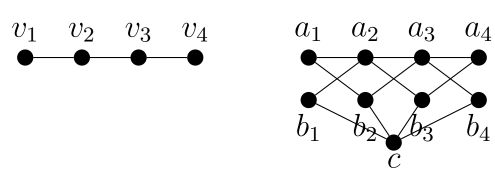
1. **(10 points)** Given that every vertex of a finite simple graph G has degree of 2 or more, explain why it must be the case that G contains a cycle.
2. **(10 points)** A simple graph is called r-*regular* if every vertex has degree r. Suppose G is a non-empty r-regular simple graph for $r \geq 2$ which does not have C_3 or C_4 as a subgraph. Show that it must be the case that $|G| \geq r^2 + 1$.
3. **(5 point bonus)** Let G be a simple graph which contains a cycle C, and a path of length k between two vertices of C. Show that G must contain a cycle of length at least $\lceil \sqrt{k} \rceil$.
4. **(10 points)** The n-cube Q^n is a graph in which each vertex is labeled with an n-bit binary sequence, and in which two vertices are adjacent if their labels differ in exactly one bit. For instance, $Q^1 = K_2$, $Q^2 = C_4$, and Q^3 is pictured below.



   (a) **(5 points)** By explicitly describing a Hamiltonian cycle, show that the n-cube is Hamiltonian for all n.

   (b) **(5 points)** Either by explicitly describing an Eulerian tour/showing that one doesn't exist, or by appealing to an alternative criterion for a graph to be Eulerian, determine which n-cubes are Eulerian.

5. **(10 points)** Show that if a simple connected planar graph G with $\|G\| \geq 2$ contains no subgraph isomorphic to C_3 (i.e. contains no triangles), then it must be the case that $\|G\| \leq 2|G| - 4$.
6. **(10 points)** Suppose the complete graph K_r is a subgraph of simple graph G, and that the maximum degree of any vertex in G is Δ. Explain why $r \leq \chi(G) \leq \Delta + 1$.
7. **(10 points)** Give an example of a connected graph with no cycles, and an ordering of the vertices thereof, such that a greedy coloring would require 4 colors. Show that your example can actually be colored with 2 colors.
8. **(5 point bonus)** For G with n vertices $v_1, v_2, \ldots, v_n$, let $M(G)$ be a graph with $2n+1$ vertices $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n, c$ such that, for v_i adjacent to v_j, it is the case that a_i is adjacent to a_j and b_j, and a_j is adjacent to b_i; furthermore, c is adjacent to every b_i. For example, below is an illustration of P_4 and $M(P_4)$:



   Show that $\chi(M(G))$ as always exactly equal to $\chi(G) + 1$.

> At the other end of the spectrum is, for example, graph theory, where the basic object, a graph, can be immediately comprehended. One will not get anywhere in graph theory by sitting in an armchair and trying to understand graphs better. Neither is it particularly necessary to read much of the literature before tackling a problem: it is of course helpful to be aware of some of the most important techniques, but the interesting problems tend to be open precisely because the established techniques cannot easily be applied.
> —W.T. Gowers

due April 9, 2012