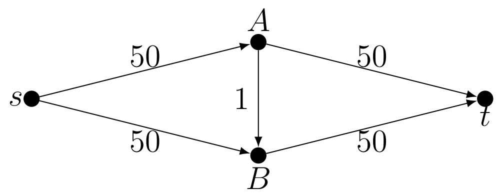
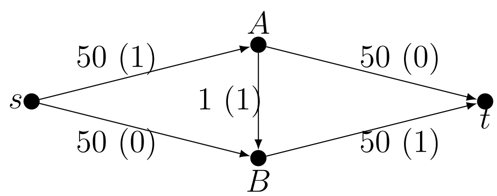
1. **(10 points)** *Demonstrate that, with sufficiently poorly-chosen flow-augmentations, the following graph might take as many as 100 iterations of the Ford-Fulkerson algorithm to find a maximum flow. Also show that, with well-chosen flow augmentations, a maximum flow can be found with only two iterations.*

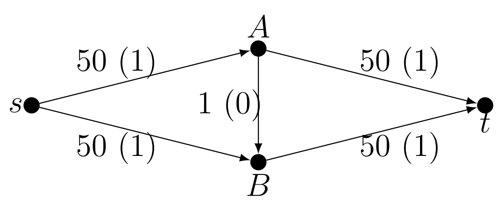


Let us consider the effect of sequentially augmenting the path $s \to A \to B \to t$ and $s \to B \leftarrow A \to t$.

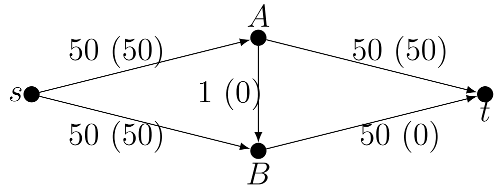
Our digraph is initialized to zero flow, so the edges $s \to A$, $A \to B$, and $B \to t$ initially have unused capacity 50, 1, and 50 respectively; thus augmentation on this path is constrained by the bottleneck $A \to B$ to be only one unit of flow, yielding the following flow:



Now, augmenting along the path $s \to B \leftarrow A \to t$, we see that while both $s \to A$ and $A \to t$ admit additional flow of 50, the backflow $B \leftarrow A$ only admits *one* unit of flow reduction, resulting in the flow:

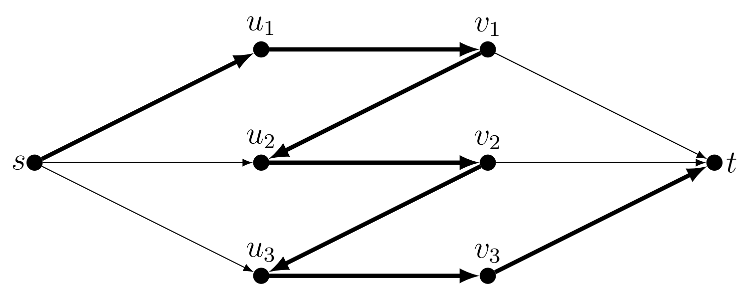


and if we now perform augmentation on $s \to A \to B \to t$, we will once again route a single unit of flow through $A \to B$, which we will subtract again with a successive augmentation along $s \to B \leftarrow A \to t$. This pair of augmentations in sequence increases total value by 2 with each performance, so it will take 50 repititions of these two augmentations (i.e. 100 augmentations in all) to reach the optimal flow



Note that this flow could have been reached much more quickly with "intelligent" augmentation choice (what is meant by "intelligent" shall be left unclear; there are methods whereby a "good" augmenting path can be chosen, but they're outside the scope of our study): if we choose to augment $s \to A \to t$ and $s \to B \to t$, we can reach this optimal flow in only two steps.

2. **(10 points)** *Find a digraph and a flow thereon which could not be improved by naïve flow expansion (i.e. simply attempting to add flow along some path), but which has value only one-third (or less) of the maximum possible flow.*
   A simple such example appears below; it can be generalized to produce arbitrarily bad flows. Every edge has capacity 1, and capacities are not displayed for simplicity. The utilized edges (which are used to full capacity) are drawn in heavier lines:

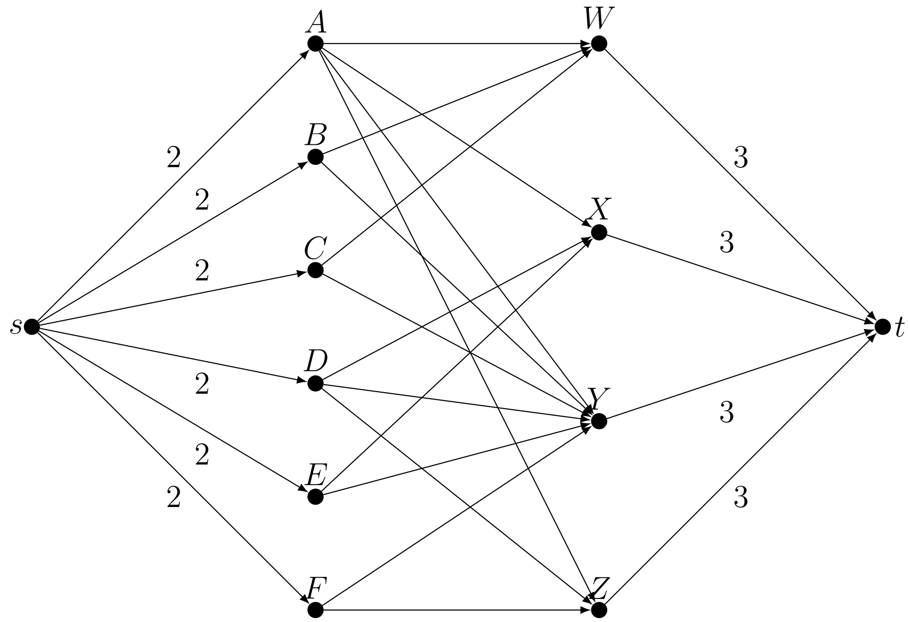


   No naïve flow augmentation is possible, since there are no under-capacitiy edges between the u_i and v_i vertices (note that there *are* several Ford-Fulkerson augmentations possible, such as $s \to u_2 \leftarrow v_1 \to t$). This flow has a value of 1, but a value-3 flow could be constructed on this graph easily by using each of the paths $s \to u_i \to v_i \to t$.
   By adding additional vertex-pairs and switchbacks, it is easy to extend this into an example where a naïve-maximal flow is only an nth of the actual maximum flow for any value of n.

3. **(10 points)** *6 people (A, B, C, D, E, and F) are to be assigned to 4 committees (W, X, Y, and Z). Each person can serve on no more than 2 committees, and each committee should have 3 people on it. Below is a table indicating which people are eligible to serve on which committees:*

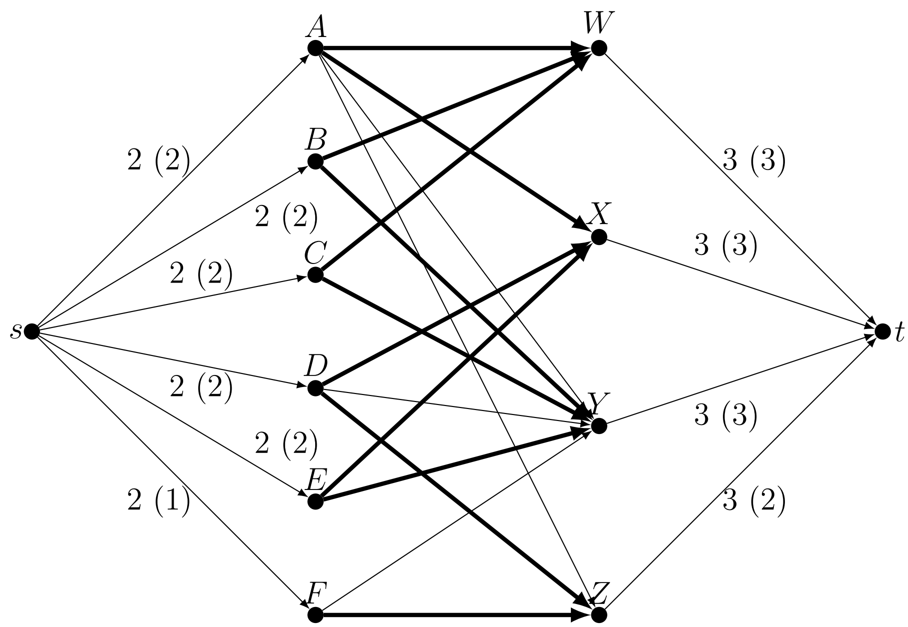
	W	X	Y	Z
A	$\surd$	$\surd$	$\surd$	$\surd$
B	$\surd$		$\surd$	
C	$\surd$		$\surd$	
D		$\surd$	$\surd$	$\surd$
E		$\surd$	$\surd$	
F			$\surd$	$\surd$

   (a) **(5 points)** *Produce a digraph with capacities whose flows would represent committee assignments.*
   This is an assignment problem, so we have flows from individuals to assignments, from the source to individuals, and from the assignments to the sink. Since each person's committee-service fills one gap, the edges from individuals to assignments have capacity 1; since each person can serve on up to 2 committees, the capacity from the source to each person (i.e., the amount of outflow the person can possibly deliver to various assignments) is 2, while since each committee ought to have 3 members, the capacity from each committee to the sink (i.e., the amount of inflow the committee can possibly receive from various individuals) is 3, and flow from individuals to committees represents assignment to that committee. The digraph is depicted below, with edges of capacity 1 left unlabeled for simplicity:



(b) **(5 points)** *Find a maximum flow on your digraph. What does this flow's value tell you about your committee assignment?*

A maximum flow is exhibited below; for simplicity, edges of capacity 1 are unlabeled, and are drawn with heavier lines to indicate flow and lighter lines if not used:



This flow is not unique and there are many other flows of equal value. That this flow is of maximum value can be easily seen by attempting to find a Ford-Fulkerson augmenting path; such a path would need to start with the step along the underutilized edge $s \to F$, and then along the underutilized edge $F \to Y$, and then along a backflow edge $Y \leftarrow E$, $Y \leftarrow C$, or $Y \leftarrow B$, but from there no further progress can be made, so there is no augmenting path.

The value of this flow is 11, which is less than we would expect with a sound committee assignment, in which each of the 6 individuals and 4 committees see full service with a total of 12 assignments. However, maximality of the flow assures us that our failure to assign everyone perfectly was not a result of foolishly suboptimal assignment on our part: there is in fact *no* flow of value 12 and this flow, which leaves committee Z unfilled and member F underutilized, is the best we can do under our constraints.

4. **(10 points)** *You are building circular bracelets with 6 beads on them; you have beads in red, yellow, and green. You want to have at least one bead of each color on every bracelet, and two bracelets are considered to be identical if one can be produced by flipping or rotating the other. How many different bracelets are possible?*

We may consider our beads as residing at the vertices of a regular hexagon for symmetry analysis-purposes. There are 12 rotation and reflection symmetries of a regular hexagon, which we can divide into six classes: the identity e; the clockwise and counterclockwise 60° rotations r and r^5; the clockwise and counterclockwise 120° rotations r^2 and r^4; the 180° rotation r^3; the three flips across edge-to-edge axes, and the three flips across vertex-to-vertex axes.

Our universe of possible colorings of a fixed hexagon is given by the surjective mappings from the vertices to the colors: using inclusion-exclusion techniques, we see that there are $3^6 - 3 \cdot 2^6 + 3 \cdot 1^6 = 540$ such. Since every coloring is invariant under the identity mapping, $\text{Inv}(e) = 540$.

In order to be invariant under r or r^5, every single vertex would need to be the same color; the number of invariants of this operation is thus the number of surejctive mappings from a set consisting of a single collection of vertices to the three colors. We could enumerate these mappings via inclusion-exclusion as $3^1 - 3 \cdot 2^1 + 3 \cdot 1^1 = 0$; however, it is equally valid to simply observe that such a surjection is impossible and that there are thus zero such; either way, $\text{Inv}(r) = \text{Inv}(r^5) = 0$.

Likewise, r^2 and r^4 map vertices in two 3-cycles, so each 3-cycle must be monochromatic to be invariant, so here we are attempting to color 2 collections of vertices surjectively with 3 colors. Again, we might either explicitly calculate $3^2 - 3 \cdot 2^2 + 3 \cdot 1^2 = 0$ or simply note the impossibility of such a surjection to determine that $\text{Inv}(r^2) = \text{Inv}(r^4) = 0$.

r^3 swaps opposite vertices, so invariance under r^3 requires that each vertex be the same color as its antipode; that is, each pair of opposite vertices must be the same color, so we are surjectively mapping 3 pairs of vertices onto the 3 colors. Here this is possible, and we might explicitly determine $3^3 - 3 \cdot 2^3 + 3 \cdot 1^3 = 6$ or note that a surjective mapping from a set to a set of the same size is a bijection, and there are 3! bijective mappings between sets of size 3; either analysis yields $\text{Inv}(r^3) = 6$.

Considering the three edge-fixing swaps, which we might call f, fr^2, and fr^4, they too map pairs of vertices onto each other, yielding the same analysis as in the previous paragraph, so $\text{Inv}(f) = \text{Inv}(fr^2) = \text{Inv}(fr^4) = 6$.

The vertex-fixing swaps, on the other hand, fix two vertices, which could be any color, and swap two pairs, each of which must be monochromatic to be invariant, so $\text{Inv}(fr) = \text{Inv}(fr^3) = \text{Inv}(fr^5) = 3^4 - 3 \cdot 2^4 + 3 \cdot 1^4 = 36$.

Now we may assemble all these invariants to find the number of classes of hexagons equivalent under these 12 transformations:

$$\frac{\sum_{g \in D_6} \text{Inv}(g)}{|D_6|} = \frac{540 + 2 \cdot 0 + 2 \cdot 0 + 6 + 3 \cdot 6 + 3 \cdot 36}{12} = 56$$

so there are 56 distinct bracelets.

5. **(5 point bonus)** *The faces of a cube are to be painted red, blue, and green; each color can be used as many times as desired or not at all. Two cube-paintings are considered to be identical if one is a rotation of the other. How many different ways are there to paint the cubes? Do not brute-force this problem!*

There are 24 rotations of a cube, which fall into 5 classes: the identity e; rotations of 90° around an axis through a pair of opposite faces (of which there are six in total: three pairs of faces, and the choice of a clockwise or counterclockwise rotation, for $3 \cdot 2 = 6$) which we shall denote r_f; rotations of 180° around an axis through a pair of opposite faces (of which there are three in total, one for each of the three pairs of faces) which we shall denote r_f^2; rotations of 60° around an axis through a pair of opposite vertices (of which there are eight in total: four pairs of vertices, and the choice of a clockwise or counterclockwise rotation, for $4 \cdot 2 = 8$) which we shall denote r_v; and rotations of 180° around an axis through a pair of opposite edges (of which there are six in total, one for each of the six pairs of edges). The invariance information of a coloring of faces of the edges is presented in the table below:

Rotation	Multiplicity	Number of cycles	Invariants
e	1	6	$3^6 = 729$
r_f	6	3	$3^3 = 27$
r_f^2	3	4	$3^4 = 81$
r_v	8	2	$3^2 = 9$
r_e	6	2	$3^3 = 27$

so using Burnside's Theorem, the number of equivalence classes of the cube under the rotation group (also called the octahedral symmetry group O) is:

$$\frac{\sum_{g \in O} \text{Inv}(g)}{|O|} = \frac{729 + 6 \cdot 27 + 3 \cdot 81 + 8 \cdot 9 + 6 \cdot 27}{24} = 57$$

so there are 57 different cube-colorings.

Life is an even-numbered problem. —Educational folk wisdom