1. **(15 points)** *You find that you need to buy 22 hats. The hat shop has as many hats as you might desire in four different varieties: stetsons, berets, stovepipes, and pillboxes. Hats within a single variety are identical.*

   (a) **(10 points)** *How many different possible ways are there for you to purchase 22 hats?*

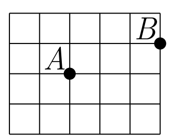
   We may consider an "order" as a placement of some number of unlabeled balls in each of four distinct boxes representing different types of hats (e.g. we could communicate an order of 10 stetsons, 3 berets, 9 stovepipes, and no pillboxes by putting 10 balls in a "stetson" box, 3 in a "beret" box, and 9 in the "stovepipe" box, while the "pillbox" box remains empty). Thus, each purchase can be associated with a distribution of 22 balls among 4 boxes. The count for doing so is a standard enumeration statistic, which we know to be $\binom{22+4-1}{4-1} = \binom{25}{3} = 2300$; alternatively, we could consider the number of ways to place 3 dividers among 22 hats, so that the hats are partitioned into 4 (possibly empty) groups, which will be declared to represent different styles of hat. This would be enumerated with $\binom{22+3}{3}$, as above.

   (b) **(5 points)** *Suppose you want to select your lot of 22 hats so that there is at least one hat of each type. How many ways are there to fulfill these instructions?*

   This situation is as in the first part of this problem, except that we constrain each box to contain at least one ball; this would be the standard enumeration statistic $\binom{22-1}{3} = \binom{21}{3} = 1932$.

   Alternatively, the above solution can be justified by pre-emptively assigning one hat of each type, leaving 18 hats left to be assigned.

2. **(10 points)** *How many direct paths which pass through at least one of the marked points are there from the lower left corner to the upper right corner of the following grid?*



We shall count the paths through point A: from the lower left corner to point A requires two steps to the right and two steps up, so this section of the path can be walked in any of $\binom{2+2}{2}$ ways. Likewise, traversal from point A to the upper right corner requires three steps to the right and two steps up, so this section of the path can be walked in any of $\binom{3+2}{2}$ ways. Since a walk through point A is built as a concatenation of these two sub-walks, the number of ways to build a path through point A is the product of the number of ways to take each of these partial walks; in other words, $\binom{2+2}{2}\binom{3+2}{2}$.

Likewise, a walk through point B begins with a walk to B, which requires five steps to the right and three steps upwards, which can be ordered in any of $\binom{5+3}{3}$ ways. The walk from B to the end is clearly unique, but a methodological purist might quantify its number as $\binom{1+0}{0}$. Since a walk through B is built from these two subwalks, the nubmer of ways to walk through B is $\binom{5+3}{3}\binom{1+0}{0}$.

However, were we to add these two possibilities, we would in fact double-count those walks which go through both points, and thus we must remove them. Walks through both points can be quantified by considering the number of walks from the beginning to A, which is $\binom{2+2}{2}$ as seen above, the number of walks from A to B, which, since such a walk requires three steps to the right and one up, can be accomplished in $\binom{3+1}{1}$ ways, and the number of walks from B to the end, which as seen above is $\binom{1+0}{0}$. We multiply these quantities to get the number of ways to do all three of these sub-walks in sequence, i.e. to walk from the beginning to the end through A and B.

Thus, the number of walks satisfying the given condition is

$$\binom{4}{2}\binom{5}{2}+\binom{8}{3}\binom{1}{0}-\binom{4}{2}\binom{4}{1}\binom{1}{0}=92.$$

3. **(15 points)** *A game is played with a fifty-card deck consisting of cards in the 5 suits of acorns, hearts, leaves, bells, and trumps, numbered 1 to 10. A hand of cards has no intrinsic order.*

   (a) **(5 points)** *How many 5-card hands are there which consist of one card in each suit, with no restrictions on numbers?*

   Our hand has no order, but individual cards may be distinguished by virtue of being different suits. Consider the card in the suit of acorns: it has 10 possible numbers. Likewise, the card in the suit of hearts has 10 possible numbers, as do the cards in leaves, bells, and trumps. A hand is thus uniquely determined by a process of 5 decisions, each of which can be resolved in 10 different ways, so there are $10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 100,000$ possible hands.

   (b) **(10 points)** *How many 5-card hands are there with two pairs (two cards in each of two different numbers, and a fifth card in a different number than either)?*

   Let us consider the features of this hand: there are two pairs, which have no intrinsic order, and one leftover card which is distinguished from these two. We must choose two distinct numbers to appear in the pairs, and since there is no intrinsic order, there are $\binom{10}{2}$ ways to do so. The number for the leftover card can be any nubmer which does not appear in the pairs – we will thus have 8 choices for this card. We may thus determine the numbers appearing in our hand any of $\binom{10}{2} \cdot 8$ ways (we could also express this as $\binom{10}{2,1,7}$ to represent choosing from a list of 10 numbers two numbers to designate as pairs, one to designate as a single, and seven to designate as not appearing in this hand).

   Now we must choose suits: Here we can distinguish between the pairs, since the pairs are distinguished by virtue of having distinct numbers chosen (e.g. we could assign suits to the lower-numbered pair, and then the higher-numbered pair). For each pair there are $\binom{5}{2}$ ways to choose suits, and there are $\binom{5}{1}$ ways to choose suits for the singleton. Thus, multiplying the nubmer of ways to choose numbers by the number of ways to choose suits, we get $\binom{10}{2}\binom{8}{1}\binom{5}{2}^2\binom{5}{1} = 180000$.

4. **(10 points)** *Let a_1, a_2, a_3, a_4, a_5, and a_6 be integers. Prove that there is a nonempty sum (possibly consisting of a single element) of the form $a_i + a_{i+1} + a_{i+2} + \cdots + a_j$ which is divisible by 6.*

Let us consider the seven values 0, a_1, $a_1 + a_2$, $a_1 + a_2 + a_3$, $\ldots$, $a_1 + a_2 + a_3 + a_4 + a_5 + a_6$ and classify them into the modular congruence classes modulo 6; since there are seven numbers here and six congruence classes, two of them are in the same class, and thus for some $i < j$, it is the case that

$$a_1 + a_2 + a_3 + \cdots + a_{i-1} \equiv a_1 + a_2 + a_3 + \cdots + a_j \pmod 6$$

and thus, subtracting the two sides, it is the case that $a_i + a_{i+1} + a_{i+2} + \cdots + a_j$ is congruent to zero modulo 6 — or in other words, is divisible by 6.

5. **(20 points)** *Answer the following questions about set unions.*

   (a) **(10 points)** *Find the number of functions from* $\{1, 2, \ldots, 8\}$ *to* $\{1, 2, 3, 4, 5, 6, 7\}$ *so that every even number in the range (i.e. 2, 4, and 6) is mapped to by at least one element of the domain.*

   Let X be a set containing every function from $\{1, 2, \ldots, 8\}$ to $\{1, 2, 3, 4, 5, 6, 7\}$; let A_2, A_4, and A_6 be sets respectively containing those functions which do not map any elements of their domains to 2, 4, and 6. Clearly $|X| = 7^8$, since a function can be constructed by mapping each element of the domain to any of 7 different values. Similarly, $|A_2| = |A_4| = |A_6| = 6^8$, since functions lying in any of those sets only have 6 possible values to which each element of the domain can be mapped. Likewise, $|A_2 \cap A_4| = |A_2 \cap A_6| = |A_4 \cap A_6| = 5^8$ and $|A_2 \cap A_4 \cap A_6| = 4^8$.

   The functions we wish to enumerate are those not lying in any of A_2, A_4, or A_6, so we specifically wish to find $|X - (A_2 \cup A_4 \cup A_6)|$, and by inclusion-exclusion:

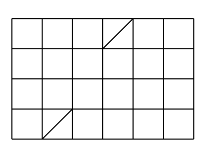
$$
\begin{aligned}
&|X - (A_2 \cup A_4 \cup A_6)| \\
&\quad = |X| - |A_2| - |A_4| - |A_6| + |A_2 \cap A_4| + |A_2 \cap A_6| + |A_4 \cap A_6| - |A_2 \cap A_4 \cap A_6| \\
&\quad = 7^8 - 3 \cdot 6^8 + 3 \cdot 5^8 - 4^8 = 1832292
\end{aligned}
$$

   (b) **(10 points)** *If* $|A_1| = 30$, $|A_2| = 12$, *and* $|A_3| = 8$ *what is the largest that* $|A_1 \cup A_2 \cup A_3|$ *can be? What is the smallest it can be? Under what conditions does each of these two possibilities occur?*

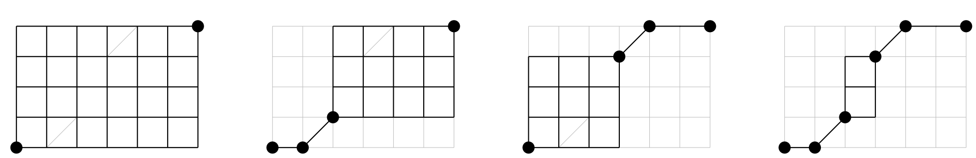
   Inclusion-exclusion suggests that this set has size no larger than $|A_1| + |A_2| + |A_3|$, since this is, if anything, an overcount from which the pairwise intersections will end up subtracted. This value is achievable when the intersections are themselves empty, so that the remaining terms in the inclusion-exclusion are all zero. In such a case, $|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| = 50$.

   The smallest this set could be, logically, is $|A_1|$. Since $A_1 \subseteq A_1 \cup A_2 \cup A_3$, we know that $A_1 \cup A_2 \cup A_3$ is at least as large as A_1, which has size 30. This case actually occurs if A_2 and A_3 are both subsets of A_1.

6. **(5 point bonus)** *How many paths consisting of steps to the north, east, and northeast along the shown paths are possible from the southwest to northeast corner of the following grid?*



We can solve this problem by counting disjoint groups of paths classified by their diagonal-edge utilization. Below are four subgrids of the given grid, with indicated waypoints necessary to visit:



The first grid has a classic orthogonal traversal,which can be completed in any of $\binom{10}{4}$ ways. The second has constrained movement on the first two steps, and $\binom{7}{3}$ ways to proceed thereafter. The third has $\binom{6}{3}$ paths, and the fourth $\binom{3}{1}$ paths, so when we add up all these cases there are

$$\binom{10}{4} + \binom{7}{3} + \binom{6}{3} + \binom{3}{1} = 268 \text{ paths.}$$