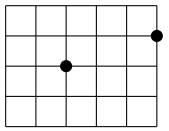
For full credit show all of your work (legibly!), unless otherwise specified. Answers need not (and probably should not) be completely reduced unless otherwise stated, and may be left in terms of sums, differences, products, quotients, exponentials, factorials, and binomial coefficients.

1. **(15 points)** You find that you need to buy 22 hats. The hat shop has as many hats as you might desire in four different varieties: stetsons, berets, stovepipes, and pillboxes. Hats within a single variety are identical.

   (a) **(10 points)** One example of a hat purchase would be: 10 stetsons, 3 berets, 9 stovepipes, and no pillboxes. How many different possible ways are there for you to purchase 22 hats? (Hint: reframe this with a balls-in-boxes paradigm)

   (b) **(5 points)** Suppose you want to select your lot of 22 hats so that there is at least one hat of each type. How many ways are there to fulfill these instructions?

2. **(10 points)** How many direct paths which pass through at least one of the marked points are there from the lower left corner to the upper right corner of the following grid?



1	/15
2	/10
3	/15
4	/10
5	/20
6	/(5)
Σ	/70

3. **(15 points)** A game is played with a fifty-card deck consisting of cards in the 5 suits of acorns, hearts, leaves, bells, and trumps, numbered 1 to 10. A hand of cards has no intrinsic order.

   (a) **(5 points)** How many 5-card hands are there which consist of one card in each suit, with no restrictions on numbers? (Example hand: the 4 of trumps, the 2 of hearts, the 7 of leaves, the 4 of acorns, and the 10 of bells)

   (b) **(10 points)** How many 5-card hands are there with two pairs (two cards in each of two different numbers, and a fifth card in a different number than either)?

4. **(10 points)** Let a_1, a_2, a_3, a_4, a_5, and a_6 be integers. Prove that there is a nonempty sum (possibly consisting of a single element) of the form $a_i + a_{i+1} + a_{i+2} + \cdots + a_j$ which is divisible by 6. (Hint: consider the seven values 0, a_1, $a_1 + a_2$, $a_1 + a_2 + a_3$, $\ldots$, $a_1 + a_2 + a_3 + a_4 + a_5 + a_6$; what can you say about them?)

5. **(20 points)** Answer the following questions about set unions.

   (a) **(10 points)** Find the number of functions from $\{1, 2, \ldots, 8\}$ to $\{1, 2, 3, 4, 5, 6, 7\}$ so that every even number in the range (i.e. 2, 4, and 6) is mapped to by at least one element of the domain.

   (b) **(10 points)** If $|A_1| = 30$, $|A_2| = 12$, and $|A_3| = 8$ what is the largest that $|A_1 \cup A_2 \cup A_3|$ can be? What is the smallest it can be? Under what conditions does each of these two possibilities occur?

6. **(5 point bonus)** How many paths consisting of steps to the north, east, and northeast along the shown paths are possible from the southwest to northeast corner of the following grid?