1. **(15 points)** *Computationally, a vector is simply a list of numbers. We may represent an n-dimensional vector $\vec{a}$ as a list of n coordinates $(a_1, a_2, a_3, \ldots, a_n)$.*

   (a) **(10 points)** *Write an algorithm, using only simple computational steps, to compute the dot product of the vectors $\vec{a}$ and $\vec{b}$. Recall that a dot product of two vectors is the sum of the coordinatewise products, e.g.* $(5, 3, 1, -2) \cdot (-1, 0, 4, 3) = 5 \cdot -1 + 3 \cdot 0 + 1 \cdot 4 + (-2) \cdot 3 = -7$.

   Input: sequences $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$
   Output: number c
   set c **to** 0;
   set i **to** 0;
   while $i \leq n$ **do**
   | **set** c **to** $c + a_i b_i$;
   | **set** i **to** $i + 1$;
   return determined value c;

   (b) **(5 points)** *Justify and state a good asymptotic bound in big-O notation on the number of steps taken by your algorithm.*

   The innermost set of instructions (adding a product to c) takes constant time, or $O(1)$ time. However, this instruction is repeated when i is 1, when i is 2, when i is 3, and so forth up to when i is n, so the time taken by performing this set of instructions n times is $n \cdot O(1)$, or the linear time $O(n)$.

2. **(10 points)** *Find the closed form of the recurrence relation given by initial conditions $a_0 = 5$, $a_1 = 0$, and $a_n = 2a_{n-1} + 24a_{n-2}$ for $n \geq 2$.*

   Letting $a_n = \lambda^n$ yields the equation $\lambda^n = 2\lambda^{n-1} + 24\lambda^{n-2}$; dividing by λ^{n-2} gives $\lambda^2 = 2\lambda + 24$, which has solutions 6 and -4, so $a_n = 6^n$ and $a_n = (-4)^n$ are both solutions to the recurrence (but not to the initial conditions; thus the general solution of the recurrence is $a_n = k \cdot 6^n + \ell(-4)^n$. To satisfy the initial conditions, it must be the case that $5 = k \cdot 6^0 + \ell(-4)^0$ and $0 = k \cdot 6^1 + \ell(-4)^1$; the solution of this pair of simultaneous equations is $k = 2$, $\ell = 3$, so the closed form for a_n is $2 \cdot 6^n + 3(-4)^n$.

3. **(15 points)** *Anna, Béla, Charles, Diane, and Edgar have dug up a treasure chest full of identical gold coins which they will share among themselves. According to their particular piratical code, Anna is to be given at least 10 coins, Béla and Charles are each to get either 5 or 6 coins (they could each receive the same or different numbers), Diane may receive any number of coins, and Edgar must receive at least 4 coins. Let a_n represent the number of ways in which n coins might be distributed.*

   (a) **(5 points)** *Find a formula for the ordinary generating function $\sum_{n=0}^{\infty} a_n z^n$.*

   Since any number of coins 10 or greater can be given to Anna (and each number of coins can only be given to her in one way), the function describing the process of providing her with coins is $1z^{10} + 1z^{11} + 1z^{12} + \cdots = \frac{z^{10}}{1-z}$. Likewise, Béla and Charles each can be provided with coins in one of two ways, represented by the polynomial $1z^5 + 1z^6$, Diane's associated function is $1z^0 + 1z^1 + 1z^2 + \cdots = \frac{1}{1-z}$, and Edgar's is $1z^4 + 1z^5 + 1z^6 + \cdots = \frac{z^4}{1-z}$, so the generating function for the distribution as a whole is the product:

$$\frac{z^{10}}{1-z}(z^5 + z^6)^2 \frac{1}{1-z} \cdot \frac{z^4}{1-z}$$

(b) **(15 points)** *Either using the ordinary generating function or by other means, determine how many ways there are to share a chest of* 30 *coins. You need not arithmetically simplify your answer.*

We algebraically simplify the generating function above to get

$$\sum_{n=0}^{\infty} a_n z^n = \frac{z^{26} + 2z^{25} + z^{24}}{(1-z)^3}$$
$$= (z^{26} + 2z^{25} + z^{24}) \sum_{n=0}^{\infty} \binom{n+2}{2} z^n$$

so in order to find a_{30}, we identify the coefficient of z^{30} on the right side; there are three addends contributing towards the z^{30} term, which are $z^{26}\binom{6}{2}z^4$, $2z^{25}\binom{7}{2}z^5$, and $z^{24}\binom{8}{2}z^6$, so that

$$a_{30} = \binom{6}{2} + 2\binom{7}{2} + \binom{8}{2} = 85$$

Alternatively, we could get the same answer via direct enumerative techniques, but they may be more difficult.

4. **(10 points)** *Find the following generating functions:*

(a) **(5 points)** *Let a_n be the number of ways to place n* distinct *objects in 5 boxes so that each box contains fewer than 4 items. Determine the formula for the exponential generating function $\sum_{n=0}^{\infty} a_n \frac{z^n}{n!}$.*

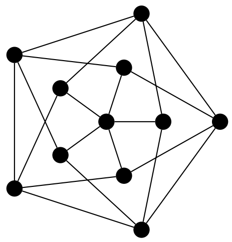
Each box has one way of containing 0, 1, 2, or 3 items, so each box is associated with the exponential generating function $\frac{1z^0}{0!} + \frac{1z^1}{1!} + \frac{1z^2}{2!} + \frac{1z^3}{3!}$. To get the generating function of the distribution to 5 boxes, we multiply their five associated generating functions, yielding $\left(\frac{1z^0}{0!} + \frac{1z^1}{1!} + \frac{1z^2}{2!} + \frac{1z^3}{3!}\right)^5$.

(b) **(5 points)** *Let b_n be the number of ways to partition n into a sum of the numbers* 1, 3, *and* 4. *Determine the formula for the ordinary generating function $\sum_{n=0}^{\infty} b_n z^n$.*

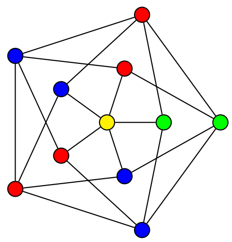
We may use no 1s, one 1, two 1s, three 1s, or so forth, each in exactly one way. These choices correspond respectively to contributions of 1, 2, 3, and so forth towards the sum, so the selection of how many 1s to use has associated generating function $1z^0 + 1z^1 + 1z^2 + \cdots = \frac{1}{1-z}$.

Likewise, we may use no 3s, one 3, two 3s, three 3s, or so forth, each in exactly one way. These choices correspond respectively to contributions of 3, 6, 9, and so forth towards the sum, so the selection of how many 3s to use has associated generating function $1z^0 + 1z^3 + 1z^6 + \cdots = \frac{1}{1-z^3}$. A similar argument yields selection function $\frac{1}{1-z^4}$ for the selection of how many 4s to use. Assembling the generating function for the partition by multiplication of the generating functions for the three individual choices involved in creating a partition, we get $\frac{1}{(1-z)(1-z^4)(1-z^4)}$.

5. **(20 points+5 point bonus)** *Let G be the graph shown below; label vertices as necessary.*



(a) **(10 points)** *Demonstrate via an explicit coloring that* $\chi(G) \leq 4$*, and give an argument that* $\chi(G) > 2$*.*



The above picture demonstrates that G is 4-colorable, and thus has chromatic number of no more than 4. In order for G to be 2-colorable, it would have to be bipartite, which is to say, it would need to have no odd cycles as subgraphs, but note that the exterior vertices are joined in a cycle of length 5. Thus G is not bipartite and therefore has chromatic number larger than 2.

(b) **(5 points)** *Is this graph Eulerian? Explain why or why not.*

(c) **(5 points)** *Demonstrate that this graph has a subgraph isomorphic to* C_6*.*

(d) **(5 point bonus)** *Is this graph planar? Either give an explicit planar representation or explain your reasoning.*