7.1.2 *Construct a generating function for a_n, the number of distributions of n identical juggling balls to*

1. *Six different jugglers with at most four balls distributed to each juggler.*

   We want our exponent on z to record the number of juggling balls distributed by every step of our process. Let the individual steps be the act of giving balls to a single juggler. Since our balls are identical, our choices in a single step are: give no balls to the juggler (which can be done only one way), represented by the polynomial $1z^0$ (or just 1); give one ball (which can be done in only one way), represented by the polynomial $1z^1$, and so forth up to giving 4 balls to a juggler. Thus, the act of giving a juggler up to 4 balls is represented by the polynomial $1+z+z^2+z^3+z^4$. Doing so for each of six jugglers, we get $(1+z+z^2+z^3+z^4)^6$ as our generating function.

2. *Five different jugglers with between three and seven balls (inclusive) distributed to each juggler.*

   As above, but the act of giving each juggler a collection of balls is rtepresented by the polynomial $z^3+z^4+z^5+z^6+z^7$; so doing so for each of five jugglers has generating function $(z^3+z^4+z^5+z^6+z^7)^5$.

7.1.5 *Find a generating function for the number of integers whose digits sum to n among*

1. *Integers from 0 to 9999.*

   An easy representation for these is as four-digit number-sequences with leading zeroes allowed; so we represent 37 as 0037, for instance (note that this has no effect on the digit sum). We break our selection of a number down into the four steps of picking individual digits. Since the information we most want to record with each step is the digit sum, we use the value of a digit as the exponent of z. Thus, for each digit, we have ten choices zero through nine, and for each choice, we want to record the number of ways to make that choice (of which there is only one) as the coefficient, and the value of that choice as the exponent. We thus associate with choice of a digit the polynomial $1z^0+1z^1+1z^2+\cdots+1z^9$; doing so four times, we get the generating function $(1+z+z^2+\cdots+z^9)^4$.

2. *Four-digit integers.*

   This is as above, but with the restriction that our first digit cannot be zero; thus the first-digit selection polynomial omits the $1z^0$ term, giving us the generating function $(z+z^2+\cdots+z^9)(1+z+z^2+\cdots+z^9)^3$.

7.1.6 *Find a generating function for the number of ways to select n balls from an infinite supply of red, white, and blue balls subject to the constraint that the number of blue balls selected is at least 3, the number of red balls selected is at most 4, and an odd number of white balls are selected.*

Since we want to record the number of balls selected, we shall use that as the exponent of z. Our selection can be broken up into 3 independent procedures: selection of red balls, white balls, and blue balls. Our selection of red balls demands no more than 4, so there are 5 possibilities: we can select zero, one, two, three, or four. This step is thus representable by the polynomial $1 + z + z^2 + z^3 + z^4$. The selection of white balls demands an odd number, so we may select one ball, or three, or five, and so forth: this yields the infinite series $z^1 + z^3 + z^5 + z^7 + \cdots$, since there is no constraint placed on how many white balls we take. Likewise, we may take any number of blue balls three or higher: we may take three, or four, or five, and so forth, yielding the series $z^3 + z^4 + z^5 + \cdots$ to describe this step. Multiplying all our steps gives us the generating function for the process as a whole:

$$(1 + z + z^2 + z^3 + z^4)(z + z^3 + z^5 + z^7 + \cdots)(z^3 + z^4 + z^5 + \cdots)$$

7.1.14 *Find a generating function for the number of positive integer solutions of*

1. $2x_1 + 3x_2 + 4x_3 + 5x_4 = n$.

   Let the exponent of z represent the total sum on the left side of the equation; then choice of x_1 as 1, 2, 3, and so forth would increase the left side by 2, or 4, or 6, and so forth, so the contribution of x_1-selection to the generating function is $z^2 + z^4 + z^6 + z^8 + \cdots$. Likewise selecting x_2 to be a positive integer increases our total by thrice its value, so selection of x_2 may yield an increase of any multiple of 3 in the total left-side sum, so x_2-selection is associated with the series $z^3 + z^6 + z^9 + z^{12} + \cdots$. Likewise, x_4 and x_5 are associated with the respective series $z^4 + z^8 + z^{12} + z^{16} + \cdots$ and $z^5 + z^{10} + z^{15} + z^{20} + \cdots$.

2. $x_1 + x_2 + x_3 + x_4 = n$, *where each* x_i *satisfies* $2 \le x_i \le 5$.

   Here we can freely choose x_1 to be 2, 3, 4, or 5, which results in addition of 2, 3, 4, or 5 to the total sum being kept track of; thus, our decision procedure for x_1 is represented by the polynomial $z^2 + z^3 + z^4 + z^5$; likewise for x_2, x_3, and x_4, so our generating function is $(z^2 + z^3 + z^4 + z^5)$.

7.2.1. 1. *Find the coefficient of* z^k *in* $(z^4 + z^5 + z^6 + z^7 + \cdots)^5$, $k \ge 20$.

We use the known series expansion $\frac{1}{(1-z)^\ell} = \sum_{n=0}^{\infty} \binom{n+\ell-1}{\ell-1} z^n$ below:

$$\begin{aligned}(z^4 + z^5 + z^6 + z^7 + \cdots)^5 &= (z^4)^5(1 + z + z^2 + z^3 + \cdots)^5 \\ &= z^{20}\left(\frac{1}{1-z}\right)^5 \\ &= z^{20}\sum_{n=0}^{\infty}\binom{n+5-1}{5-1}z^n \\ &= \sum_{n=0}^{\infty}\binom{n+4}{4}z^{n+20}\end{aligned}$$

This is an acceptable but not entirely canonical representation of the generating function: to be standardized (and useful for answering the question asked) we'd like to change the exponent on z to be a single variable. So let $k = n + 20$, replacing every n above with $k - 20$:

$$\sum_{k-20=0}^{\infty} \binom{(k-20)+4}{4} z^k = \sum_{k=20}^{\infty} \binom{k-16}{4} z^k$$

so the coefficient of z^k is $\binom{k-16}{4}$ for $z \geq 20$ (and for $z < 20$, the coefficient is zero, as these terms do not appear in the series expansion).

2. *Find the coefficient of z^k in $(z + z^3 + z^5)(1 + z)^n$, $k \geq 5$.*
We use the known binomial expansion $(1+z)^n = \sum_{i=0}^{\infty} \binom{n}{i} z^i$ below. Note that the upper bound can be ∞, since every binomial coefficient $\binom{n}{k}$ with $k > n$ is zero; this is not necessary, but actually makes our series manipulation easier, since we need not worry about the upper limit of the sum:

$$\begin{aligned}(z + z^3 + z^5)(1 + z)^n &= (z + z^3 + z^5) \sum_{i=0}^{\infty} \binom{n}{i} z^i \\ &= \sum_{i=0}^{\infty} \binom{n}{i} z^{i+1} + \sum_{i=0}^{\infty} \binom{n}{i} z^{i+3} + \sum_{i=0}^{\infty} \binom{n}{i} z^{i+5}\end{aligned}$$

Now we need to perform index-shifts on each sum to get our z-terms to match; so we introduce new indices $k = i+1$, $k = i+3$, and $k = i+5$ to the three sums:

$$\begin{aligned}&\sum_{i=0}^{\infty} \binom{n}{i} z^{i+1} + \sum_{i=0}^{\infty} \binom{n}{i} z^{i+3} + \sum_{i=0}^{\infty} \binom{n}{i} z^{i+5} \\ &= \sum_{k=1}^{\infty} \binom{n}{k-1} z^k + \sum_{k=3}^{\infty} \binom{n}{k-3} z^k + \sum_{k=5}^{\infty} \binom{n}{k-5} z^k \\ &= z + nz^2 + \left[\binom{n}{2} + 1\right] z^3 + \left[\binom{n}{3} + n\right] z^4 + \sum_{k=5}^{\infty} \left[\binom{n}{k-1} + \binom{n}{k-3} + \binom{n}{k-3}\right] z^k\end{aligned}$$

So despite all the special-casing necessary for $k < 5$ (which need not be calculated as explicitly as shown here), we see that the coefficient of z^k for $k \geq 5$ will always be $\binom{n}{k-1} + \binom{n}{k-3} + \binom{n}{k-3}$.

7.4.6. *Find an exponential generating function for the number of ways to distribute n different objects to six different jugglers if each juggler receives between three and five objects.*

One can think of an assignment of distinct objects to n people as an n-ary sequence of digits; for instance, in this case the senary digit-string "1043552" would be an assignment of 7 distinct objects, giving object #2 to person zero, object #1 to person one, object #7 to person two, and so forth up to giving objects #5 and #6 to person five. As such, we can think of this selection process as a commingled selection of how many zeroes there are in the string, how many ones, etc. Each must appear between three and five times, so the exponential generating functions representing each individual selection is

$$1\left(\frac{z^3}{3!}\right)+1\left(\frac{z^4}{4!}\right)+1\left(\frac{z^5}{5!}\right)=\frac{z^3}{6}+\frac{z^4}{24}+\frac{z^5}{120}$$

And the generating function for the process as a whole consists of six such processes, thus $\left(\frac{z^3}{6}+\frac{z^4}{24}+\frac{z^5}{120}\right)^6$. The expansion of this form is difficult to calculate and rather unilluminating. For example, the initial term is $\frac{z^{18}}{46656}$, more canonically represented as $137225088000\frac{z^{18}}{18!}$, which demonstrates that there are approximately 137 trillion distributions of a mere 18 objects under this scheme.

7.4.10. *Find the number of strings of length n that can be constructed using the alphabet $\{a,b,c,d,e\}$ if:*

1. *b occurs an odd number of times.*
   Free selection of any number of instances of a particular letter yields generating function $1\left(\frac{z^0}{0!}\right)+1\left(\frac{z^1}{1!}\right)+1\left(\frac{z^2}{2!}\right)+1\left(\frac{z^3}{3!}\right)+\cdots=e^z$; selection of an odd number of instances of a particular letter yields $1\left(\frac{z^1}{1!}\right)+1\left(\frac{z^3}{3!}\right)+1\left(\frac{z^5}{5!}\right)+1\left(\frac{z^7}{7!}\right)+\cdots=\frac{e^z-e^{-z}}{2}$, as seen in problem 7.4.4. Thus, given exponential selection functions for all five letters which we want to mingle, we see that the generating function for this process as a whole is

$$\begin{aligned}(e^z)^4\left(\frac{e^z-e^{-z}}{2}\right)&=\frac{e^{5z}-e^{3z}}{2}\\&=\frac{1}{2}\left(\sum_{n=0}^{\infty}\frac{(5z)^n}{n!}-\sum_{n=0}^{\infty}\frac{(3z)^n}{n!}\right)\\&=\sum_{n=0}^{\infty}\frac{5^n-3^n}{2}\frac{z^n}{n!}\end{aligned}$$

   so the number of ways to achieve an arrangement of n letters is the coefficient of $\frac{z^n}{n!}$, seen above to be $\frac{5^n-3^n}{2}$.
2. *both a and b occur an odd number of times.* Using the same processes in the previous section, we get

$$\begin{aligned}(e^z)^3\left(\frac{e^z-e^{-z}}{2}\right)^2&=\frac{e^{5z}-2e^{3z}+e^z}{4}\\&=\frac{1}{4}\left(\sum_{n=0}^{\infty}\frac{(5z)^n}{n!}-2\sum_{n=0}^{\infty}\frac{(3z)^n}{n!}+\sum_{n=0}^{\infty}\frac{z^n}{n!}\right)\\&=\sum_{n=0}^{\infty}\frac{5^n-2\cdot 3^n+1}{4}\frac{z^n}{n!}\end{aligned}$$

   so as above, the coefficient of $\frac{z^n}{n!}$ gives us our desired formula, $\frac{5^n-2\cdot3^n+1}{4}$.

8.2.4. 1. *Find the specific solution of* $a_n = 2a_{n-1} + 15a_{n-2}$ *with initial conditions* $a_0 = 1$ *and* $a_1 = 2$.

The characteristic polynomial of this recurrence relation is $x^2 - 2x - 15$, which has roots -3 and 5, so the general solution to this recurrence is $a_n = k_1(-3)^n + k_2 5^n$. Now, we plug in the initial values $n = 0$ and $n = 1$ to this general solution to find that, for our initial conditions to be met, it must be the case that:

$$\begin{cases} 1 = k_1 + k_2 \\ 2 = -3k_1 + 5k_2 \end{cases}$$

This system can be shown, using linear algebra, back-solution, or any other system-of-equations solution methods, to have solution $k_1 = \frac{3}{8}$, $k_2 = \frac{5}{8}$. Thus, the particular solution to the recurrence relation meeting the given initial conditions is

$$a_n = \frac{3(-3)^n + 5(5^n)}{8}$$

2. *Find the specific solution of* $a_n = 3a_{n-1} + 2a_{n-2}$ *with initial conditions* $a_0 = 1$ *and* $a_1 = 2$.

The characteristic polynomial of this recurrence relation is $x^2 - 3x - 2$, which has roots $\frac{3+\sqrt{17}}{2}$ and $\frac{3-\sqrt{17}}{2}$, so the general solution to this recurrence is $a_n = k_1\left(\frac{3+\sqrt{17}}{2}\right)^n + k_2\left(\frac{3-\sqrt{17}}{2}\right)^n$. Now, we plug in the initial values $n = 0$ and $n = 1$ to this general solution to find that, for our initial conditions to be met, it must be the case that:

$$\begin{cases} 1 = k_1 + k_2 \\ 2 = \frac{3+\sqrt{17}}{2}k_1 + \frac{3-\sqrt{17}}{2}k_2 \end{cases}$$

This system can be shown, using linear algebra, back-solution, or any other system-of-equations solution methods, to have solution $k_1 = \frac{1}{2} + \frac{\sqrt{17}}{34}$, $k_2 = \frac{1}{2} - \frac{\sqrt{17}}{34}$. Thus, the particular solution to the recurrence relation meeting the given initial conditions is

$$a_n = \left(\frac{1}{2} + \frac{\sqrt{17}}{34}\right)\left(\frac{3+\sqrt{17}}{2}\right)^n + \left(\frac{1}{2} - \frac{\sqrt{17}}{34}\right)\left(\frac{3-\sqrt{17}}{2}\right)^n$$

8.3.2. *Find the general solution of*

1. $a_n = a_{n-1} + 6a_{n-2} + 2n$.

The associated linear homogeneous recurrence relation is $b_n = b_{n-1} + 6b_{n-2}$, which has characteristic polynomial $x^2 - x - 6$ with roots 3 and -2, so the general solution of the associated homogeneous polynomial is $b_n = k_1 3^n + k_2(-2)^n$.

Now, we must find a particular solution to the original nonhomogeneous recurrence relation. Since the nonhomogeneous component is a linear polynomial in n, we choose a template for the particular solution which is also a linear polynomial: $a_n^* = \ell_0 + \ell_1 n$. Since this does not overlap our homogeneous solution in any of its terms, we can be confident that it will suffice. Substituting the definition of a_n^* in for a_n in the recurrence, we get the equation

$$\begin{aligned} \ell_0 + \ell_1 n &= (\ell_0 + \ell_1(n-1)) + 6(\ell_0 + \ell_1(n-2)) + 2n \\ \ell_0 + \ell_1 n &= (7\ell_0 - 13\ell_1) + (7\ell_1 + 2)n \end{aligned}$$

Matching up coefficients, we thus have the system of equations:

$$\begin{cases} \ell_0 = 7\ell_0 - 13\ell_1 \\ \ell_1 = 7\ell_1 + 2 \end{cases}$$

which we can solve to find that $\ell_1 = -\frac{1}{3}$ and $\ell_0 = -\frac{13}{18}$. Thus, $a_n^* = -\frac{1}{3}n - \frac{13}{18}$ is a solution to this recurrence; to get the general solution, we add in the general homogeneous solution to get:

$$a_n = a_n^* + b_n = -\frac{1}{3}n - \frac{13}{18} + k_1 3^n + k_2(-2)^n$$

2. $a_n - 5a_{n-1} + 6a_{n-2} = 5^n$.

   The associated linear homogeneous recurrence relation is $b_n - 5b_{n-1} + 6b_{n-2} = 0$, which has characteristic polynomial $x^2 - 5x + 6$ with roots 3 and 2, so the general solution of the associated homogeneous polynomial is $b_n = k_1 3^n + k_2 2^n$.

   Now, we must find a particular solution to the original nonhomogeneous recurrence relation. Since the nonhomogeneous component is an exponential expression, we choose a template for the particular solution which is also an exponential expression: $a_n^* = \ell 5^n$. Since this does not overlap our homogeneous solution in any of its terms, we can be confident that it will suffice. Substituting the definition of a_n^* in for a_n in the recurrence, we get the equation

$$\begin{aligned} \ell 5^n - 5\ell 5^{n-1} + 6\ell 5^{n-2} &= 5^n \\ 25\ell 5^{n-2} - 25\ell 5^{n-2} + 6\ell 5^{n-2} &= 25 \cdot 5^{n-2} \\ 6\ell &= 25 \\ \ell &= \frac{25}{6} \end{aligned}$$

   Thus, $a_n^* = \frac{25}{6}5^n$ is a solution to this recurrence; to get the general solution, we add in the general homogeneous solution to get:

$$a_n = a_n^* + b_n = \frac{25}{6}5^n + k_1 3^n + k_2 2^n$$

4.1.6. *Suppose we have two words of arbitrary length. If a computer can compare any two letters, describe an algorithm for determining which word comes first in alphabetical order.*

```
Input: words a_1 a_2 a_3 ... a_n and b_1 b_2 b_3 ... b_m
set i to 1;
repeat
    if i > n and i > m then decide that a = b;
    if i > n and i ≤ m then decide that a < b;
    if i ≤ n and i > m then decide that a > b;
    if i ≤ n and i ≤ m then
        if a_i < b_i then decide that a < b;
        if a_i > b_i then decide that a > b;
        if a_i = b_i then set i to i + 1 and make no decision;

until we have made a judgment ;
```

This algorithm is an expression of how we would compare words. The process is simply a matter of comparing the first letters (a_1 to b_1), and returning the result of that comparison if unequal, and moving onto the next letter if impossible. i records which letter we are on at each stage of the process, so it starts at 1, since we start by comparing the first letters, and increments when a letter fails to provide an adequate comparison. The conditions at the beginning indicate what to do when we run out of letters from one or the other string. If we run out on one, it means that string is shorter (e.g. "ant" is lexicographically smaller than "anteater"); if we run out on both, it means we've exhausted both words without finding a way in which they differ, in which case the two words are identical.