1. **(22 points)** *Below, a "number" is any string of digits that does not begin with a zero.*

   (a) **(2 points)** *How many 6-digit numbers are there?*

   We may select the first digit in any of 9 ways (any digit from 1–9), and the remaining five digits in any of 10 ways (any digit from 0–9). Since a six-digit number is produced by making all these individual choices, the multiplication principle indicates that the number of ways to select a six-digit number is $9 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 900000$.

   Alternatively, you could note that the six-digit nubmers are the integers between 100000 and 999999, of which there are $999999 - 100000 + 1 = 900000$.

   (b) **(6 points)** *How many 6-digit numbers are there in which at least one digit is even?*

   From our result above, we will subtract out all those in which no digits are even (which is to say, those in which all the digits are odd). Such "bad numbers" can be selected by choosing any of five digits (1, 3, 5, 7, or 9) for each of the six positions in the number, so there are $5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5$ numbers we must exclude from our count, yielding $9 \cdot 10^5 - 5^6$, which would be difficult to simplify without a calculator, but which is equal to 884375.

   (c) **(6 points)** *How many 6-digit numbers are there in which at least one digit is even and no two digits are the same?*

   We start by counting those 6-digit numbers in which no two digits are the same. The first digit, as seen in part (a), can be chosen in any of nine ways (any number from 1–9). The second digit can be chosen in any of nine ways as well (any number from 0–9, except that chosen for the first). Since we now have two digits which have been used, the third digit can be chosen in any of 8 ways, the fourth in any of 7 ways, the fifth in any of 6 ways, and the sixth in any of 5 ways, so there are $9 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5$ six-digit numbers with no repeated digits.

   Now we would need to subtract out, as in part (b), those in which all digits are odd. Fortunately, though, this set is empty, since there are only five even digits, so it would be impossible for any number with six distinct digits to have only odd digits, so this subtraction step is unnecessary (or, alternatively, involves subtracting zero). Thus our answer is $9 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5$, which, although difficult to compute by hand, is equal to 136080.

   (d) **(8 points)** *Determine a generating function for a_n, the number of 6-digit numbers in which n of the digits are even. The generating function need not be algebraically simplified.*

   The first digit could be any of 1, 3, 5, 7, or 9, contributing zero to our count of even digits, or it could be 2, 4, 6, or 8, contributing one towards our count of even digits; thus the generating function for the first digit alone is $(5z^0 + 4z^1)$. The second and subsequent digits are similar, except that they have 5 instead of 4 ways to be even, and thus have associated generating functions $(5z^0 + 5z^1)$, so the generating function for the 6-digit number as a whole, since it is the product of the generating functions associated with each of its digits, is $(5 + 4z)(5 + 5z)^5$.

2. **(10 points)** *Consider the following algorithm performed on a sequence of numbers $a_1, a_2, \ldots, a_n$.*

   (1) *Let $i = 1$.*

   (2) *Let $q = i$ and let $j = q + 1$.*

   (3) *If $a_j < a_q$, then let $q = j$.*

(4) Increment j.

(5) If $j \leq n$, then return to step (3).

(6) Swap the values of a_i and a_q (if $i = q$, do nothing).

(7) Increment i.

(8) If $i < n$, then return to step (2).

(a) **(4 points)** *Walk through the algorithm's procedure when performed on the 5-term sequence* $(4, 8, 1, 10, 2)$*. What does this algorithm seem to do?*

Steps 2–5 probe each number from i to n, setting q equal to whichever index has the smallest associated value a_q. So, for example, in the first step, when $i = 1$, q would be set to 3, since a_3 is the smallest element of $a_1, \ldots, a_5$. Then that would be swapped to position a_1, so the first time we reach step 7, the sequence would have been modified to $(1, 8, 4, 10, 2)$.

On the second passthrough, when $i = 2$, we would probe from a_2 to a_5 looking for the smallest element; now q would be set to 5, since a_5 is small. So we would swap that to position 2, yielding the sequence $(1, 2, 4, 10, 8)$.

When $i = 3$, we probe a_3, a_4, and a_5 for the smallest; now it is a_3, so we would do nothing.

When $i = 4$, we look at a_4 and a_5; a_5 is smaller, so it is swapped with a_4 to yield $(1, 2, 4, 8, 10)$.

The apparent result of this procedure is to sort our sequence. This is indeed the function of this algorithm, which is known as selection sort. It's a particularly useful sort if our data is for some reason "immobile", since it only swaps when it knows exactly where a number should go, but as we shall see in the next part of this problem, it is not terribly efficient in other ways.

(b) **(6 points)** *Give a big-O estimate of the number of operations, in terms of n, which this algorithm takes to perform its task.*

For each value of i, steps 2–5 will be performed $n - i$ times, since the procedure looks at all values between $i + 1$ and n when seeking the smallest index. The entirety of the procedure will be performed $n - 1$ times: once with $i = 1$, once with $i = 2$, and so forth up to $i = n - 1$, so the total number of cycles through steps 2–5 will be

$$(n-1) + (n-2) + (n-3) + \cdots + 2 + 1 = \frac{n(n-1)}{2} = O(n^2)$$

3. **(18 points)** *Answer the following questions about recurrence relations.*

(a) **(6 points)** *Find the general solution to the recurrence relation $a_n = 4a_{n-1} + 21a_{n-2}$.*

This is a homogeneous linear recurrence, so its fundamental solutions will be of the form λ^n. Substituting this into the above equation, we get $\lambda^n = 4\lambda^{n-1} + 21\lambda^{n-2}$, which, on dividing by λ^{n-2}, yields the quadratic $\lambda^2 = 4\lambda + 21$, which has solutions $\lambda = 7, -3$. so the general solution is the linear combination $a_n = k \cdot 7^n + \ell(-3)^n$.

(b) **(12 points)** *Find the particular solution to the recurrence relation $b_n = 4b_{n-1} + 21b_{n-2} - 25 \cdot 2^n$ with initial conditions $b_0 = 1$ and $b_1 = 27$.*

We shall start by finding a solution to the inhomogeneous equation here, and then shall combine it with the homogeneous general solution above; finally we will plug in the initial values to get the constants in the specific solution.

Since the inhomogeneous part is a multiple of 2^n, one solution is likewise a multiple of 2^n: let $b_n = c \cdot 2^n$ be a solution to the inhomogeneous equation. Then:

$$\begin{aligned} c \cdot 2^n &= 4c \cdot 2^{n-1} + 21c \cdot 2^{n-2} - 25 \cdot 2^n \\ c \cdot 2^2 &= 4c \cdot 2^1 + 21c - 25 \cdot 2^2 \\ (4 - 8 - 21)c &= -100 \\ c &= 4 \end{aligned}$$

so one solution to the inhomogeneous equation is $b_n = 4 \cdot 2^n$; note this does not match our initial conditions, however. The *general* solution to the inhomogeneous equation is then $b_n = k \cdot 7^n + \ell(-3)^n + 4 \cdot 2^n$, integrating the general terms from part (a). Using the initial conditions, we can find the particular solution:

$$\begin{cases} 1 = b_0 = k \cdot 7^0 + \ell(-3)^0 + 4 \cdot 2^0 \\ 27 = b_1 = k \cdot 7^1 + \ell(-3)^1 + 4 \cdot 2^1 \end{cases}$$

which simplifies to

$$\begin{cases} -3 = k + \ell \\ 19 = 7k - 3\ell \end{cases}$$

which can be solved to give $k = 1$ and $\ell = -4$, so the final recurrence is $b_n = 7^n - 4(-3)^n + 4 \cdot 2^n$.

4. **(24 points)** *We are placing objects in 4 boxes, and each box must receive at least one item.*

   (a) **(6 points)** *If we have exactly six identical items, and our boxes are distinguishable, how many ways are there to distribute the items?*

   We pre-emptively place one item in each of the boxes, leaving 2. We can now place the remaining two objects in any of the boxes: we may either use the balls-and-walls statistic to note that there are $binom2 + 32 = 10$ such placements, or consider two cases: the two balls are in the same box, which can be done in any of 4 ways, or in separate boxes, any of 6 ways.

   (b) **(6 points)** *If we have exactly six distinguishable items, and our boxes are distinguishable, how many ways are there to distribute the items? Your answer probably should not be arithmetically simplified.*

   This problem is exactly identical to the number of surjective mappnigs from a set of size six to a set of size four, which we can solve via inclusion-exclusion. We start by considering *all* possible placements of objects in boxes, of which there are 4^6. We now must subtract out those which leave a box empty: we select one box to leave empty, and distribute the objects among the other three boxes, which we can do in $\binom{4}{1}3^6$ ways. However, now we have doubly-removed those cases where two boxes are empty, and must add them back in: we select two boxes, and then fill the remaining boxes, which we can do in $\binom{4}{2}2^6$ ways. Lastly, we have now over-reincluded the case where three boxes are empt, and must subtract it back out: there are $\binom{4}{3}1^6$ such. Thus, our total is:

$$4^6 - \binom{4}{1}3^6 + \binom{4}{2}2^6 - \binom{4}{3}1^6$$

which is equal to 1560, although calculating such by hand is difficult.

(c) **(8 points)** *Find an exponential generating function describing the number of ways to distribute n distinguishable objects among 4 distinguishable boxes with no box left empty.*

Each box can receive 1 or more objects, so each box has an associated exponential generating function $\frac{1}{1!}z^1 + \frac{1}{1!}z^2 + \frac{1}{1!}z^3 + \cdots = e^z - 1$. Since the generating function for the process as a whole is the product of the generating functions of its constituent parts, the process has generating function $(e^z - 1)^4$.

(d) **(4 points)** *Explain, without explicit arithmetic computation, why your answer to part (b) must be divisible by 24.*

Since each box contains at least one item and the items are distinct, boxes are in fact characterized by their contents. Thus, each distribution belongs to a "family" of 24 distinct distributions yielded by swapping around the labels on the boxes; alternatively, we could note that the statistic describing surjective placements of distinct objects into distinct boxes is 24 times the statistic describing surjective placements of distinct objects into indistinct boxes; since both of these are integers, the former must be divisible by 24.

5. **(24 points)** *We are concocting a pitcher of a refreshing beverage with a single measuring cup (so an integer number of cups of each ingredient is used), and combining orange juice, ginger ale, pineapple juice, and iced tea. Our particular taste preferences call for at least one cup of each beverage to be used, for at least three cups of orange juice to be used, and for no more than 6 cups of iced tea to be used. Let a_n be the number of possible n-cup mixes we could make.*

(a) **(8 points)** *Find a formula for the generating function $\sum_{n=0}^{\infty} a_n z^n$.*

Orange juice contributes at least 3 cups but with no maximum limitation, so it has an associated generating function of $z^3 + z^4 + \cdots = \frac{z^3}{1-z}$. Iced tea provides at least 1 but no more than 6 cups, so it is associated with the generating function $z + z^2 + \cdots + z^6 = \frac{z - z^7}{1-z}$. The other two ingredients contribute at least one cup but no maximum, for associated generating function $z + z^2 + \cdots = \frac{z}{1-z}$. Assembled, these give $\frac{z^6 - z^{12}}{(1-z)^4}$

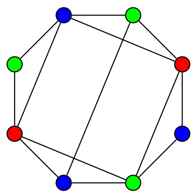
(b) **(16 points)** *If we have a 16-cup pitcher, how many different mixes could we make to fill it? You may use your generating function to solve this problem if desired.*

We wish to find the coefficient of z^{16} in our generating function. Note that

$$\frac{z^6 - z^{12}}{(1-z)^4} = (z^6 - z^{12}) \sum_{n=0}^{\infty} \binom{n+3}{3} z^n$$

which will contribute the terms $z^6 \cdot \binom{13}{3} z^{10}$ and $-z^{12} \cdot \binom{7}{3} z^4$, so our answer is $\binom{13}{3} - \binom{7}{3}$.

6. **(12 points)** *Let G be the graph illustrated below. Answer the following questions. You may label the original graph, if desired.*



(a) **(5 points)** *Is this graph Eulerian? Why or why not?*

Two of the vertices have odd degree, so this graph is non-Eulerian.

(b) **(7 points)** *Demonstrate that $\chi(G) = 3$.*

Since the graph contains odd cycles, it has chromatic number greater than 2. That 3 colors are sufficient is depicted on the graph above.

7. **(15 points)** *We have a flagpole which can fly nine miniature flags, one above another. We have four red flags, two blue flags, two black flags, and one white flag.*

(a) **(4 points)** *How many ways are there to arrange these nine flags on the pole?*

This is a pure multinomial coefficient: we have 4 objects of one type, 2 of each of 2 other types, and one unique object to arrange, so our total number of arrangements is

$$\binom{9}{4,2,2,1} = \binom{9}{4}\binom{5}{2}\binom{3}{2}\binom{1}{1} = \frac{9!}{4!2!2!1!}$$

This is 3780, which would be difficult to calculate by hand.

(b) **(6 points)** *How many ways are there to arrange these nine flags if the two blue flags must not be next to each other?*

From our above statistic, we will subtract those where the two blue flags *are* adjacent: considering the two blue flags as a single entity, there are $\binom{8}{4,2,1,1}$ arrangements to be subtracted off, giving a result of

$$\binom{9}{4,2,2,1} - \binom{8}{4,2,1,1}$$

which is 2940, but that is difficult to calculate by hand.

(c) **(5 points)** *How many ways are there to arrange these nine flags if the two blue flags must not be next to each other, and additionally the two black flags must not be next to each other?*

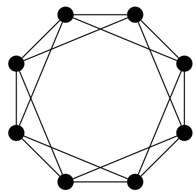
As seen above, we will subtract out the nubmer of arrangements in which the blue flags are adjacent, of which there are $\binom{8}{4,2,1,1}$; additionally we will subtract out those where the two black flags are adjacent, of which there are also $\binom{8}{4,2,1,1}$; however, now we have doubly-removed the cases where both are true, and must re-insert them; these cases, where the black and blue flags are both monolithic, are counted by the multinomial $\binom{7}{4,1,1,1}$, so our total is

$$\binom{9}{4,2,2,1} - 2\binom{8}{4,2,1,1} + \binom{7}{4,1,1,1}$$

which is 2310.

8. **(8 point bonus)** *The following problems relate to forcing graphs to contain specific substructures.*

   (a) **(2 point bonus)** *Give an example of a graph on 8 vertices such that no four vertices are mutually adjacent, and no three vertices are mutually nonadjacent.*

   (b) **(6 point bonus)** *Let G be a graph such that $|G| \geq 9$. Prove that G contains either four mutually adjacent vertices or three mutually nonadjacent vertices.*

   We may specifically consider the case of $|G| = 9$; if G is larger, we could use this same proof on an arbitrary nine-vertex induced subgraph. We divide this problem into three possible cases which will describe every graph on 9 vertices.

   Case I: some vertex v has degree 4 or less. Since v has four or fewer neighbors and G contains 8 vertices ther than v, there must be four or more vertices nonadjacent to v. If any two of these four vertices are nonadjacent to each other, then v and those two vertices form a set of three mutually nonadjacent vertices. If no two of these four vertices are nonadjacent to each other, then these four vertices would all be mutually adjacent.

   Case II: some vertex v has degree 6 or more. Let us consider some neighbor w of v, and the set S of some five other vertices neighboring v; now, depending on how w relates to S, we have a few subcases.

   Case II(a): w has 3 or more neighbors in S. Let us call these neighbors a, b, and c. Now w and v are both adjacent to each other and to each of a, b, and c, so if any of a, b, and c are adjacent to each other, then that pair together with w and v forms a set of four mutually adjacent vertices. If, on the other hand, none of a, b, and c are adjacent to each other, then they collectively form a set of three mutually non-adjacent vertices.

   Case II(b): w has 2 or fewer neighbors in S. Since $|S| = 5$, we now know that w has 3 non-neighbors in S. If any two of these are non-adjacent, then these two together with w form a set of three mutually non-adjacent vertices. If, on the other hand, these 3 are all adjacent to each other, then since they are all adjacent to v, then together with v they form a set of four mutually adjacent vertices.

   Case III: every vertex has degree of exactly 5. This case is impossible, since $2\|G\| = \sum_{v \in G} \deg(G) = 9 \cdot 5 = 45$, which would require exactly 22.5 edges, an impossibility.

9. **(7 point bonus)** *The following problems relate to symmetry groups; below p will be a prime number greater than 2, and a "coloring" will be an assignment of any of n colors to each vertex of a regular p-sided polygon. Answer these questions on the back of this (or any other) sheet.*

   (a) **(3 point bonus)** *Find (with justification) a closed-form formula in terms of p and n for the number of different colorings, if two rotations of the same coloring are considered to be identical.*

There are n^p ways to color a regular polygon fixed in space; these are invariants under the identity transformation. However, if we consider *any* rotation, it will map the vertices in one enormous cycle, so the only colorings which are invariant under any rotation are the monochromatic ones, of which there are n in total. Thus, all $p-1$ rotations have n invariants, while the identity has n^p invariants, so by Burnside's theorem, the number of equivalent colorings is:

$$\frac{n^p + (p-1)n}{p}$$

(b) **(4 point bonus)** *Find (with justification) a closed-form formula in terms of p and n for the number of different colorings, if two rotations or reflections of the same coloring are considered to be identical.*

All description given above for the invariants under rotations remains true and relevant here; however now we must also take into account the flips. Since p is odd, these flips all fix a single vertex and swap the others in pairs; thus, in order to be invariant, the fixed point can be any color desired while each of the $\frac{p-1}{2}$ pairs of other vertices must be monochromatic, so we are coloring $\frac{p-1}{2} + 1$ different classes of vertices, and can do so in $n^{\frac{p+1}{2}}$ ways. Since there are p different possible flips, we use the same formula as above with an extra term and twice as many transformations in the denominator:

$$\frac{n^p + (p-1)n + n^{\frac{p+1}{2}}}{2p}$$