For full credit show all of your work (legibly!), unless otherwise specified. This exam is closed-notes and calculators may not be used. Answers need not be completely reduced unless otherwise stated, and may be left in terms of sums, differences, products, quotients, exponents, factorials, binomial coefficients, and multinomial coefficients.

1. **(22 points)** Below, a "number" is any string of digits that does *not begin with a zero.*

   (a) **(2 points)** How many 6-digit numbers are there?

   (b) **(6 points)** How many 6-digit numbers are there in which at least one digit is even?

   (c) **(6 points)** How many 6-digit numbers are there in which at least one digit is even and no two digits are the same?

   (d) **(8 points)** Determine a generating function for a_n, the number of 6-digit numbers in which n of the digits are even. The generating function need not be algebraically simplified.

1	/ 22
2	/ 10
3	/ 18
4	/ 24
5	/ 24
6	/ 12
7	/ 15
8	/ (8)
9	/ (7)
Σ	/125

2. **(10 points)** Consider the following algorithm performed on a sequence of numbers $a_1, a_2, \ldots, a_n$.

   (1) Let $i = 1$.

   (2) Let $q = i$ and let $j = q + 1$.

   (3) If $a_j < a_q$, then let $q = j$.

   (4) Increment j.

   (5) If $j \leq n$, then return to step (3).

   (6) Swap the values of a_i and a_q (if $i = q$, do nothing).

   (7) Increment i.

   (8) If $i < n$, then return to step (2).

   (a) **(4 points)** Walk through the algorithm's procedure when performed on the 5-term sequence $(4, 8, 1, 10, 2)$. What does this algorithm seem to do?

   (b) **(6 points)** Give a big-O estimate of the number of operations, in terms of n, which this algorithm takes to perform its task.

3. **(18 points)** Answer the following questions about recurrence relations.

   (a) **(6 points)** Find the general solution to the recurrence relation $a_n = 4a_{n-1} + 21a_{n-2}$.

   (b) **(12 points)** Find the particular solution to the recurrence relation $b_n = 4b_{n-1} + 21b_{n-2} - 25 \cdot 2^n$ with initial conditions $b_0 = 1$ and $b_1 = 27$.

4. **(24 points)** We are placing objects in 4 boxes, and each box must receive at least one item.

   (a) **(6 points)** If we have exactly six identical items, and our boxes are distinguishable, how many ways are there to distribute the items?

   (b) **(6 points)** If we have exactly six distinguishable items, and our boxes are distinguishable, how many ways are there to distribute the items? Your answer probably should not be arithmetically simplified.

   (c) **(8 points)** Find an exponential generating function describing the number of ways to distribute n distinguishable objects among 4 distinguishable boxes with no box left empty.

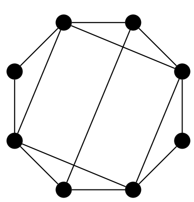
   (d) **(4 points)** Explain, without explicit arithmetic computation, why your answer to part (b) must be divisible by 24.

5. **(24 points)** We are concocting a pitcher of a refreshing beverage with a single measuring cup (so an integer number of cups of each ingredient is used), and combining orange juice, ginger ale, pineapple juice, and iced tea. Our particular taste preferences call for at least one cup of each beverage to be used, for at least three cups of orange juice to be used, and for no more than 6 cups of iced tea to be used. Let a_n be the number of possible n-cup mixes we could make.

   (a) **(8 points)** Find a formula for the generating function $\sum_{n=0}^{\infty} a_n z^n$.

   (b) **(16 points)** If we have a 16-cup pitcher, how many different mixes could we make to fill it? You may use your generating function to solve this problem if desired.

6. **(12 points)** Let G be the graph illustrated below. Answer the following questions. You may label the original graph, if desired.



   (a) **(5 points)** Is this graph Eulerian? Why or why not?

   (b) **(7 points)** Demonstrate that $\chi(G) = 3$.

7. **(15 points)** We have a flagpole which can fly nine miniature flags, one above another. We have four red flags, two blue flags, two black flags, and one white flag.

   (a) **(4 points)** How many ways are there to arrange these nine flags on the pole?

   (b) **(6 points)** How many ways are there to arrange these nine flags if the two blue flags must not be next to each other?

   (c) **(5 points)** How many ways are there to arrange these nine flags if the two blue flags must not be next to each other, and additionally the two black flags must not be next to each other?

8. **(8 point bonus)** The following problems relate to forcing graphs to contain specific substructures. Answer these questions on the back of this (or any other) sheet.

   (a) **(2 point bonus)** Give an example of a graph on 8 vertices such that no four vertices are mutually adjacent, and no three vertices are mutually nonadjacent.

   (b) **(6 point bonus)** Let G be a graph such that $|G| \geq 9$. Prove that G contains either four mutually adjacent vertices or three mutually nonadjacent vertices.

9. **(7 point bonus)** The following problems relate to symmetry groups; below p will be a prime number greater than 2, and a "coloring" will be an assignment of any of n colors to each vertex of a regular p-sided polygon. Answer these questions on the back of this (or any other) sheet.

   (a) **(3 point bonus)** Find (with justification) a closed-form formula in terms of p and n for the number of different colorings, if two rotations of the same coloring are considered to be identical.

   (b) **(4 point bonus)** Find (with justification) a closed-form formula in terms of p and n for the number of different colorings, if two rotations or reflections of the same coloring are considered to be identical.