1. **(15 points)** *A bar of iron which has been heated to 1400°F is taken from the furnace into a 100°F metalworking studio. After 5 minutes it has cooled to 800°F.*

   (a) **(5 points)** *Construct a function $T(t)$ modeling the bar's temperature t minutes after removal from the furnace.*

   We know that this problem is modeled by Newton's Law of Cooling with an ambient temperature of 100°F, so our temperature model will be $T(t) = 100 + Ce^{kt}$; it remains only to find C and k to have a final model.

   Since the bar has a temperature of 1400°F immediately upon removal from the furnace, $T(0) = 1400$. Evaluating the left side of this equation, we find that $100 + Ce^0 = 1400$; thus $C = 1300$.

   Since the bar has a temperature of 800°F five minutes later, we know that $T(5) = 800$. Expanding $T(5)$, we find that:

$$\begin{aligned} 100 + 1300e^{k \cdot 5} &= 800 \\ 1300e^{5k} &= 700 \\ e^{5k} &= \frac{7}{13} \\ 5k &= \ln \frac{7}{13} \\ k &= \frac{\ln \frac{7}{13}}{5} \end{aligned}$$

   Assembling this value of k into our equation, we find that

$$T(t) = 100 + 1300e^{\frac{\ln \frac{7}{13}}{5} t}$$

   (b) **(4 points)** *The metal can be worked as long as it is hotter than 1000°F. How soon after the bar is removed from the furnace does it become unworkable?*

   Since the metal becomes unworkable when $T(t) = 1000$, we want to find the value of t satisfying that equation:

$$\begin{aligned} 100 + 1300e^{\frac{\ln \frac{7}{13}}{5} t} &= 1000 \\ 1300e^{\frac{\ln \frac{7}{13}}{5} t} &= 900 \\ e^{\frac{\ln \frac{7}{13}}{5} t} &= \frac{9}{13} \\ \frac{\ln \frac{7}{13}}{5} t &= \ln \frac{9}{13} \\ t &= \frac{5 \ln \frac{9}{13}}{\ln \frac{7}{13}} \approx 3 \text{ minutes} \end{aligned}$$

   (c) **(6 points)** *How quickly is the bar's temperature changing immediately upon removal from the furnace?*

The time of the bar's removal from the furnace is definitionally time 0; thus the speed of the bar's cooling at that time is $T'(0)$. From the value of $T(t)$ above, we can easily compute $T'(t)$:

$$T'(t) = 1300\frac{\ln\frac{7}{13}}{5}e^{\frac{\ln\frac{7}{13}}{5}t}$$

so $T'(0) = 1300\frac{\ln\frac{7}{13}}{5}e^0 = 260\ln\frac{7}{13}$. This is approximately -161, signifying that the bar is cooling (dropping in temperature) by 161 degrees per minute.

2. **(20 points)** *You have 1200 square centimeters of material with which to make a box with a square base and an open top. Find the dimensions which maximize the volume of the box.*

   Let us denote the dimensions of the base in centimeters (which are equal, since the base is a square) as x, and let the height of the box be h centimeters. Then the box uses a total of $x^2 + 4xh$ square centimeters of material (since the base is of area x^2, and the four lateral faces are each of area xh), and has a volume of x^2h.

   Thus, our material limitations require that $x^2 + 4xh = 1200$, and our goal is to maximize x^2h. Solving the constraint for h, we find that $h = \frac{1200-x^2}{4x}$, so our volume, written solely in terms of x is the function $V(x) = x^2\left(\frac{1200-x^2}{4x}\right)$. We seek to maximize this function on the interval $(0, \sqrt{1200})$ — whose lower bound comes from the fact that x must be positive as it is a length, and whose upper bound comes from the fact that there must be at least a little left over material left to build the lateral faces, so $x^2 < 1200$.

   Solving this problem in a traditional manner, we see that $V(x) = 300x - \frac{x^3}{4}$, so $V'(x) = 300 - \frac{3x^2}{4}$.This is continuous everywhere, so itsonly critical points are where $300 - \frac{3x^2}{4} = 0$, which can be algebraically converted to $x^2 = 400$, so $x = \pm 20$.

   The critical point $x = -20$ can be rejected as being outside of the interval; the remaining critical point and the two ends of the interval are tested:

$$\lim_{x\to 0^+} V(x) = V(0) = 0$$

$$V(20) = 20^2\left(\frac{1200-400}{80}\right) = 4000$$

$$\lim_{x\to\sqrt{1200}^-} V(x) = V(\sqrt{1200}) = 0$$

   so our optimal dimensions are $20 \times 20 \times 10$.

3. **(15 points)** *The* lemniscate *is a curve satisfying the equation* $2(x^2 + y^2)^2 = 25(x^2 - y^2)$.

   (a) **(10 points)** *Find a formula for* $\frac{dy}{dx}$ *on this curve in terms of* x *and* y.

We proceed by implicit differentiation:

$$\begin{aligned}
\frac{d}{dx}\left[2(x^2+y^2)^2\right] &= \frac{d}{dx}\left[25(x^2-y^2)\right] \\
4(x^2+y^2)\frac{d}{dx}(x^2+y^2) &= 25\frac{d}{dx}(x^2-y^2) \\
4(x^2+y^2)\left(2x+2y\frac{dy}{dx}\right) &= 25\left(2x-2y\frac{dy}{dx}\right) \\
8x^3+8x^2y\frac{dy}{dx}+8xy^2+8y^3\frac{dy}{dx} &= 50x-50y\frac{dy}{dx} \\
8x^2y\frac{dy}{dx}+8y^3\frac{dy}{dx}+50y\frac{dy}{dx} &= 50x-8x^3-8xy^2 \\
\frac{dy}{dx} &= \frac{50x-8x^3-8xy^2}{8x^2y+8y^3+50y}
\end{aligned}$$

(b) **(5 points)** *Find the equation of the tangent line to the lemniscate at the point* $(3,1)$.
Using the above result, we determine that at $(3,1)$,

$$m = \frac{dy}{dx} = \frac{150-216-24}{72+8+50} = \frac{-9}{13}$$

so using the point-slope formula, we have an equation of $(y-1) = \frac{-9}{13}(x-3)$.

4. **(18 points)** *Evaluate the following integrals:*

(a) **(6 points)** $\int_0^4 x\sqrt{16-x^2}dx$.
We make use of a substitution for $16-x^2$, yielding the equation $u = 16-x^2$ and the pseudo-equation $du = -2xdx$. Thus $xdx = \frac{-1}{2}du$, which we may use to convert the entire integral, and then evaluate it:

$$\int_0^4 x\sqrt{16-x^2}dx = \int_{16}^0 -\frac{1}{2}\sqrt{u}du = -\frac{1}{3}u^{3/2}\Big]_{16}^0 = -\frac{0^{3/2}}{3}+\frac{16^{3/2}}{3} = \frac{64}{3}$$

(b) **(6 points)** $\int \frac{x^3}{x^4+1}dx$.
We shall substitute u for x^4+1, yielding the pseudo-equation $du = 4x^3du$, so we may convert our integral as such:

$$\int \frac{x^3}{x^4+1}dx = \int \frac{1}{4u}du = \frac{1}{4}\ln|u|+C = \frac{1}{4}\ln|x^4+1|+C$$

(c) **(6 points)** $\int_0^{\pi/3} \cos\theta + 2\sec\theta\tan\theta d\theta$.
This is an integral which can be directly evaluated:

$$\int_0^{\pi/3} \cos\theta+2\sec\theta\tan\theta d\theta = \sin\theta+2\sec\theta\Big]_0^{\pi/3} = \sin\frac{\pi}{3}+2\sec\frac{\pi}{3}-\sin 0-2\sec 0 = \frac{\sqrt{3}}{2}+4-0-2 = 2+\frac{\sqrt{3}}{2}$$

5. **(16 points)** *A ten-foot-long ladder is leaning against a wall, with the base of the ladder six feet from the wall. The base is sliding away from the wall at a rate of half a foot per hour.*

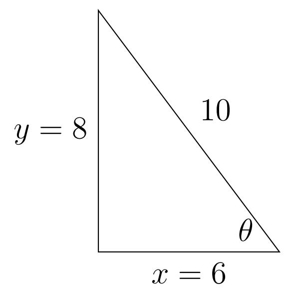
(a) **(8 points)** *How quickly is the top of the ladder slipping down the wall?*

Let us denote the height of the ladder up the wall by y, and the distance of the base from the wall by x. Under this naming scheme, the desired quantity is $\frac{dy}{dt}$. Then by the Pythagorean Theorem, $x^2 + y^2 = 10^2$ (since the length of the ladder is constant). We also know that x is currently 6, and can deduce that y is currently $\sqrt{10^2 - 6^2} = 8$. In addition, the problem tells us that $\frac{dx}{dt} = 0.5$. Thus:

$$\begin{aligned} \frac{d}{dt}(x^2 + y^2) &= \frac{d}{dt}100 \\ 2x\frac{dx}{dt} + 2y\frac{dy}{dt} &= 0 \\ \frac{dy}{dt} &= \frac{-x\frac{dx}{dt}}{y} = \frac{-3}{8} \end{aligned}$$

so the ladder is sliding down the wall at three-eighths of a foot per hour.

(b) **(8 points)** *How quickly is the angle between the ladder and the floor changing?*



Let us use the same variables as above, and additionally let θ represent the angle between the ladder and the floor, so that our quantity of interest here is $\frac{d\theta}{dt}$. Since θ is an angle of a right triangle, we can express its trig functions as ratio of the sides; the most usable formula here is $\cos\theta = \frac{x}{10}$, although others will also work. Taking the derivative of both sides, we see that

$$\frac{d\theta}{dt}(-\sin\theta) = \frac{\frac{dx}{dt}}{10}$$

and thus that

$$\frac{d\theta}{dt} = -\frac{\frac{dx}{dt}}{10}\csc\theta = \frac{-0.5}{10}\cdot\frac{10}{8} = \frac{-1}{16}$$

6. **(16 points)** *Compute the following expressions:*

(a) **(6 points)** *Compute $\frac{d}{dx}\sqrt{\arctan\sqrt{x}}$.*

In preparation for the chain rule, let us denote $\sqrt{\arctan\sqrt{x}}$ as $\sqrt{u}$, where $u = \arctan v$ and $v = \sqrt{x}$. Then

$$\frac{d}{dx}\sqrt{u} = \frac{du}{dx}\frac{1}{2\sqrt{u}} = \frac{dv}{dx}\frac{du}{dv}\frac{1}{2\sqrt{u}}$$

and evaluating all these know derivatives, we get a result of

$$\frac{1}{2\sqrt{x}}\frac{1}{1+v^2}\frac{1}{2\sqrt{u}} = \frac{1}{4\sqrt{x}(1+x)\sqrt{\arctan x}}$$

(b) **(6 points)** *Given* $f(t) = \tan \frac{e^t}{\arcsin t}$, *find* $f'(t)$.

Let $u = \frac{e^t}{\arcsin t}$; using the quotient rule we may find that $\frac{du}{dt} = \frac{\arcsin t e^t - e^t \frac{1}{\sqrt{1-t^2}}}{(\arcsin t)^2}$. Then, using the chain rule:

$$f'(t) = \frac{d}{dt} \tan u = \frac{du}{dt} \sec^2 u = \frac{\arcsin t e^t - e^t \frac{1}{\sqrt{1-t^2}}}{(\arcsin t)^2} \sec^2 \frac{e^t}{\arcsin t}$$

(c) **(4 points)** *Find* $\int \frac{d}{ds} \frac{s^3}{\sqrt{s^2+5}} ds$.
This is an antiderivative of a derivative, and it is known that doing so simply introduces a constant, so the answer is $\frac{s^3}{\sqrt{s^2+5}} + C$.

7. **(12 points)** *Consider the function* $g(x) = \frac{x}{x^2+9}$.

(a) **(5 points)** *Identify the zeroes, vertical asymptotes, and long-term behavior on both sides of this function. Clearly label which is which, and if any features are not present, say so.*

The numerator is zero (and thus the function is zero) when $x = 0$; vertical asymptotes appear when the denominator is zero, but that never happens in this case, so this function has no vertical asymptotes. As x grows very large or very negative, this function behaves like $\frac{x}{x^2} = \frac{1}{x}$, which tends towards zero in both directions.

(b) **(5 points)** *Identify the critical points of this function, and indicate whether each is a local maximum, local minimum, or neither.*

Note that $g'(x) = \frac{(x^2+9)-2x^2}{(x^2+9)^2} = \frac{9-x^2}{(x^2+9)^2}$. This exists everywhere, since the denominator is never zero, but the numerator is zero at $x = \pm 3$; thus this function has critical points at $x = \pm 3$. Since $g'(-4) = g'(4)$ is negative and $g'(0)$ is positive, we see that $x = -3$ is a minimum and $x = +3$ is a maximum.

(c) **(2 points)** *Which if any of the critical points identified above are global maxima or global minima? Show work or explain.*

Both are global extrema: this function has no asymptotes or long-term behavior that could "spil" them, so they are in fact the function's global minimum and maximum.

8. **(15 points)** *Determine the following limits.*

(a) **(5 points)** *Evaluate* $\lim_{\theta \to 0} \frac{\theta - \sin\theta}{\theta - \tan\theta}$ *or demonstrate that it cannot be evaluated.*

Since $0 - \sin 0 = 0$ and $0 = \tan 0 = 0$, this is a $\frac{0}{0}$ indeterminate form. We may thus invoke L'Hôpital's rule:

$$\lim_{\theta \to 0} \frac{\theta - \sin\theta}{\theta - \tan\theta} = \lim_{\theta \to 0} \frac{1 - \cos\theta}{1 - \sec^2\theta}$$

which is still a $\frac{0}{0}$ form, so we continue:

$$\lim_{\theta \to 0} \frac{1 - \cos\theta}{1 - \sec^2\theta} = \lim_{\theta \to 0} \frac{-\sin\theta}{-2\sec^2\theta \tan\theta} = \lim_{\theta \to 0} \cos^3 \theta 2 = \frac{1}{2}$$

(b) **(5 points)** *Evaluate* $\lim_{t \to 0} \frac{e^t - 1 - t}{t^2}$ *or demonstrate that it cannot be evaluated.*

This is a $\frac{0}{0}$ indeterminate form, so we may use L'Hôpital's rule:

$$\lim_{t \to 0} \frac{e^t - 1 - t}{t^2} = \lim_{t \to 0} \frac{e^t - 1}{2t}$$

and again, as it is still $\frac{0}{0}$:

$$\lim_{t\to 0}\frac{e^t-1}{2t}=\lim_{t\to 0}\frac{e^t}{2}=\frac{1}{2}$$

(c) **(5 points)** Using the difference quotient, *find the derivative with respect to x of the function $f(x)=20+3x-5x^2$. You may not use L'Hôpital's rule for this problem.*

$$\begin{aligned}f'(x)&=\lim_{x\to 0}\frac{f(x+h)-f(x)}{h}\\&=\lim_{x\to 0}\frac{(20+3(x+h)-5(x+h)^2)-(20+3x-5x^2)}{h}\\&=\lim_{x\to 0}\frac{3h-10xh-5h^2}{h}\\&=\lim_{x\to 0}3-10x-5h=3-10x\end{aligned}$$

9. **(15 points)** *Let $f(x)=2+2x^2-x^4$.*

(a) **(5 points)** *Where is $f(x)$ increasing? Where is it decreasing? Label which is which.*

We consider the sign-changes of $f'(x)=4x-4x^3=4x(x+1)(x-1)$. This is clearly zero at $x=-1$, $x=1$, and $x=0$. Probing between and among the zeroes indicates that when $x<-1$, $f'(x)$ is negative, when $-1<x<0$, $f'(x)$ is positive, when $0<x<1$, $f'(x)$ is negative, and when $x>1$, $f'(x)$ is positive again. Thus $f(x)$ is increasing on $(-1,0)$ and $(1,\infty)$, and is decreasing on $(-\infty,-1)$ and $(0,1)$.

(b) **(3 points)** *What are the critical points of $f(x)$? Is each a local maximum, a local minimum, or neither?*

Since $x=-1$ is a transition from decrease to increase, it is a local minimum, since $x=0$ is a transition from increase to decrease, it is a local maximum, and since $x=1$ is a transition from decrease to increase, it is a local minimum.

(c) **(7 points)** *Determine where $f(x)$ is concave up and where it is concave down, and identify points of inflection.*

We consider the sign-changes of $f''(x)=4-12x^2$. This is zero at $x=\pm\frac{1}{\sqrt{3}}$, and probing to the left and right of these zeroes indicates that $f''(x)<0$ when $|x|>\frac{1}{\sqrt{3}}$, and $f''(x)>0$ when $|x|<\frac{1}{\sqrt{3}}$. Thus, $f(x)$ is concave down on both $(-\infty,-\frac{1}{\sqrt{3}})$ and $(\frac{1}{\sqrt{3}},\infty)$, and concave up on $(-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}})$.

10. **(8 points)** *Answer the following questions about the function $f(x)=\frac{x^{2/3}}{x+1}$.*

(a) **(4 points)** *What is the domain of $f(x)$?*

The numerator is not a problem; cube roots are valid anywhere. Howver, the denominator is zero when $x=-1$, so the domain is $x\neq -1$.

(b) **(4 points)** *Where does the derivative of $f(x)$ exist?*

$f'(x)=\frac{(x+1)\cdot\frac{2}{3}x^{-1/3}-x^{2/3}}{(x+1)^2}$. This still has problems at $x=-1$, but additionally it is problematic at $x=0$, since 0 cannot be raised to a negative power. Thus, this function if differentiable only when x is neither 0 nor -1.

11. **(6 point bonus)** *Find, with explanation, a general form for the following expressions, for positive integer n:*

 (a) **(3 points)** $\frac{d^n}{dx^n}(x^2e^x)$.

 We might pre-emptively note, by using the product rule, that $\frac{d}{dx}(x^2e^x) = x^2e^x + 2xe^x$ and that $\frac{d}{dx}(xe^x) = xe^x + e^x$; additionally, we see that $\frac{d}{dx}e^x = e^x$ as we already knew. We thus reasonably expect that repeated differentiation of z^2e^x will spawn terms of the form x^2e^x, xe^x, and e^x, since only terms of those forms result from taking the derivatives of terms of those forms.

 If we perform a few simple derivatives, we can see patterns emerging.

$$\begin{aligned}
\frac{d}{dx}(x^2e^x) &= x^2e^x + 2xe^x & &= x^2e^x + 2xe^x \\
\frac{d^2}{dx^2}(x^2e^x) &= x^2e^x + 2xe^x + 2xe^x + 2e^x & &= x^2e^x + 4xe^x + 2e^x \\
\frac{d^3}{dx^3}(x^2e^x) &= x^2e^x + 2xe^x + 4xe^x + 4e^x + 2e^x & &= x^2e^x + 6xe^x + (2+4)e^x \\
\frac{d^4}{dx^4}(x^2e^x) &= x^2e^x + 2xe^x + 6xe^x + 6e^x + (2+4)e^x & &= x^2e^x + 8xe^x + (2+4+6)e^x
\end{aligned}$$

 and we could form a conjecture (which is not too hard to prove, although it would be easiest to do using the principle of induction, which is not covered in this class), that

$$\frac{d^n}{dx^n}(x^2e^x) = x^2e^x + nxe^x + (2 + 4 + \cdots + 2(n-1))e^x$$

 and, using rules for simplifying arithmetic series, the last term can actually be folded into a closed form to yield:

$$\frac{d^n}{dx^n}(x^2e^x) = x^2e^x + nxe^x + n(n-1)e^x$$

 (b) **(3 points)** $\lim_{x\to\infty} \frac{x^n}{e^{0.01x}}$.

 This is a $\frac{\infty}{\infty}$ form, and will remain one for n successive invocations of L'Hôpital's rule! Afterwards, however, we will find that

$$\lim_{x\to\infty} \frac{x^n}{e^{0.01x}} = \lim_{x\to\infty} \frac{n(n-1)(n-2)(n-3)\cdots(2)(1)}{(0.01)(0.01)\cdots(0.01)e^{0.01x}} = \lim_{x\to\infty} \frac{n!}{(0.01)^n e^{0.01x}} = 0$$