1. **(20 points)** *Answer the following derivative-related questions.*

   (a) **(8 points)** *Calculate* $\frac{d}{dx}\sin(\ln(\arctan x))$.

   Initial analysis suggests that we will be using the chain rule twice (since this expression is the sine of a complicated expression, which is itself the logarithm of a complicated expression). We shall define $v = \arctan x$ and $u = \ln v$, and then:

$$\begin{aligned}\frac{d}{dx}\sin(\ln(\arctan x)) &= \frac{d}{dx}\sin u \\ &= \frac{du}{dx}\frac{d}{du}\sin u \\ &= \left(\frac{d}{dx}\ln v\right)\cos u \\ &= \left(\frac{dv}{dx}\frac{d}{dv}\ln v\right)\cos u \\ &= \left(\frac{d}{dx}\arctan x\right)\cdot\frac{1}{v}\cdot\cos u \\ &= \frac{1}{1+x^2}\cdot\frac{1}{v}\cdot\cos u = \frac{\cos(\ln(\arctan x))}{(1+x^2)\arctan x}\end{aligned}$$

   (b) **(6 points)** *If* $f(q) = \frac{\ln q}{q^4}$, *find* $f'(q)$.

   Since we are taking a derivative of a quotient, we may approach this question with the quotient rule:

$$f'(q) = \frac{q^4\left(\frac{d}{dq}\ln q\right) - \ln q\frac{d}{dq}q^4}{(q^4)^2} = \frac{q^4\cdot\frac{1}{q} - \ln q\cdot 4q^3}{(q^4)^2} = \frac{1-4\ln q}{q^5}$$

   The last step is a simplification and is not necessary when solving this problem.

   (c) **(6 points)** *If* $y = \arcsin(3x)\tan x$, *find* $\frac{dy}{dx}$.

   This is a derivative of a product, which will require the product rule; there is also a chain rule within one of the factors, but that can be done implicitly rather than by a direct invocation of the chain rule.

$$\frac{dy}{dx} = \left(\frac{d}{dx}\arcsin(3x)\right)\tan x + \arcsin 3x\frac{d}{dx}\tan x = 3\frac{1}{\sqrt{1-(3x)^2}}\tan x + \arcsin 3x\sec^2 x$$

2. **(10 points)** *If* $y = \csc\frac{t^3-3t}{e^t+\sin t}$, *find* $\frac{dy}{dt}$.

   This expression will require the use of the chain rule (since on its outermost layer it is a cosecant of a complicated expression) and then the quotient rule (since that complicated expression is itself a quotient). We let $u = \frac{t^3-3t}{e^t+\sin t}$ and proceed to invoke these two rules as necessary:

$$\begin{aligned}
\frac{dy}{dt} &= \frac{d}{dt}\csc u \\
&= \frac{du}{dt}\frac{d}{du}\csc u \\
&= \left(\frac{d}{dt}\frac{t^3-3t}{e^t+\sin t}\right)(-\csc u \cot u) \\
&= \frac{(e^t+\sin t)\frac{d}{dt}(t^3-3t) - (t^3-3t)\frac{d}{dt}(e^t+\sin t)}{(e^t+\sin t)^2}(-\csc u \cot u) \\
&= \frac{(e^t+\sin t)(3t^2-3) - (t^3-3t)(e^t+\cos t)}{(e^t+\sin t)^2}\left(-\csc\frac{t^3-3t}{e^t+\sin t}\cot\frac{t^3-3t}{e^t+\sin t}\right)
\end{aligned}$$

3. **(10 points)** *Calculate* $\frac{d}{ds}\big(\cos(s)\tan(s^2-3s)\big)$.

This expression will require the use of the product rule (since on its outermost layer it is a product of two factors) and then the chain rule (since one of those factors is the tangent of a complicated expression). We let $u = s^2 - 3s$ and proceed to invoke these two rules as necessary:

$$\begin{aligned}
\frac{d}{ds}\big(\cos(s)\tan(s^2-3s)\big) &= \frac{d}{ds}\big(\cos(s)\tan(u)\big) \\
&= \left(\frac{d}{ds}\cos(s)\right)\tan(u) + \cos(s)\left(\frac{d}{ds}\tan(u)\right) \\
&= -\sin(s)\tan(u) + \cos(s)\left(\frac{du}{ds}\frac{d}{du}\tan(u)\right) \\
&= -\sin(s)\tan(u) + \cos(s)\left(\frac{d}{ds}s^2-3s\right)\sec^2(u) \\
&= -\sin(s)\tan(u) + \cos(s)(2s-3)\sec^2(u) \\
&= -\sin(s)\tan(s^2-3s) + \cos(s)(2s-3)\sec^2(s^2-3s)
\end{aligned}$$

4. **(12 points)** *A pure 150-gram sample of the radioactive material Cobalt-Thorium-G is taken to a shielded laboratory for testing. 6 days later it is found that, due to radioactive decay, only 125 grams of CoTh-G remain.*

   (a) **(5 points)** *Produce a function $f(t)$ modeling the quantity of Cobalt-Thorium-G left in the sample after t days.*

   The known model for radioactive decay is an exponential decay function, so we know that our funtion should be $f(t) = Ce^{kt}$ for constants C and k (we know k will be negative, because this is a decay function, but that information is not of immediate use to us). Furthermore, the described information about the material quantities at various times tells us that $f(0) = 150$ (as the initial sample is 150g) and $f(6) = 125$ (since six days later the sample has decayed to 125g). Thus we may easily determine C:

$$f(0) = 150 \quad\Rightarrow\quad Ce^0 = 150 \quad\Rightarrow\quad C = 150$$

and with a little more work, may determine k:

$$\begin{aligned} f(6) &= 125 \\ 150e^{k\cdot 6} &= 125 \\ e^{6k} &= \frac{125}{150} = \frac{5}{6} \\ 6k &= \ln\frac{5}{6} \\ k &= \frac{\ln\frac{5}{6}}{6} \end{aligned}$$

so our function should be $f(t) = 150e^{\frac{\ln(5/6)}{6}t}$.

(b) **(4 points)** *How many days will it take the sample to decay to half of its original mass?*

The sample will have halved in mass when $f(t) = 75$, so we want to find the value of t satisfying that equation:

$$\begin{aligned} 150e^{\frac{\ln(5/6)}{6}t} &= 75 \\ e^{\frac{\ln(5/6)}{6}t} &= \frac{75}{150} = \frac{1}{2} \\ \frac{\ln(5/6)}{6}t &= \ln\frac{1}{2} \\ \frac{\ln(5/6)}{6}t &= \frac{6\ln\frac{1}{2}}{\ln(5/6)} \approx 22.81 \text{ days} \end{aligned}$$

The last approximation is given for illumination and could not easily be calculated under exam conditions.

(c) **(3 points)** *How quickly, in grams per day, is the sample decaying 3 days into the experiment?*

The rate of decay at time $t = 3$ is definitionally $f'(3)$. From the formula for $f(t)$ above, we can easily compute $f'(t)$:

$$f'(t) = 150 \cdot \ln(5/6)6e^{\frac{\ln(5/6)}{6}t}$$

so $f'(3) = 150 \cdot \ln(5/6)6e^{\frac{\ln(5/6)}{2}}$. This is approximately -4.16, signifying that the sample is losing a mass of -4.16 grams per day.

5. **(15 points)** *The cissoid of Diocles is a curve satisfying the equation $x(x^2 + y^2) = 4y^2$.*

(a) **(12 points)** *Find a formula for $\frac{dy}{dx}$ on this curve.*

Taking the derivative of each side, we get

$$\begin{aligned}
\frac{d}{dx}\left(x(x^2+y^2)\right) &= \frac{d}{dx}\left(4y^2\right) \\
(x^2+y^2) + x\frac{d}{dx}(x^2+y^2) &= \frac{dy}{dx}\frac{d}{dy}\left(4y^2\right) \\
(x^2+y^2) + 2x^2 + x\frac{d}{dx}y^2 &= \frac{dy}{dx}8y \\
3x^2 + y^2 + x\frac{dy}{dx}\frac{d}{dy}y^2 &= \frac{dy}{dx}8y \\
3x^2 + y^2 + 2xy\frac{dy}{dx} &= 8y\frac{dy}{dx} \\
2xy\frac{dy}{dx} - 8y\frac{dy}{dx} &= -3x^2 - y^2 \\
(2xy - 8y)\frac{dy}{dx} &= -3x^2 - y^2 \\
\frac{dy}{dx} &= \frac{-3x^2 - y^2}{2xy - 8y} = \frac{3x^2 + y^2}{8y - 2xy}
\end{aligned}$$

(b) **(3 points)** *Find the equation of the tangent line to the curve at* $(2, -2)$.

At the specific values $x = 2$, $y = -2$, we see from the above that

$$\frac{dy}{dx} = \frac{3 \cdot 2^2 + (-2)^2}{8(-2) - 2 \cdot 2(-2)} = \frac{16}{-8} = -2$$

Thus we have a line of the form $y = -2x + b$; plugging in $(2, -2)$ we can solve for b:

$$\begin{aligned}
-2 &= -2 \cdot 2 + b \\
2 &= b
\end{aligned}$$

so $y = -2x + 2$ is the tangent line.

6. **(8 points)** *Estimate the following values using appropriate linear approximations.*

(a) **(4 points)** 1.03^5.

We consider the function $f(x) = x^5$, whose derivative is $f'(x) = 5x^4$. For x close to 1 (as 1.03 is), we can use the linear approximation:

$$f(x) \approx f(1) + (x - 1)f'(1)$$

Since $f(1) = 1$ and $f'(1) = 5$, it follows that

$$f(1.03) \approx 1 + 0.03(5) = 1.15$$

For purposes of comparison, the actual value of 1.03^5 is 1.1592740743.

(b) **(4 points)** $\sqrt{99.8}$.

We consider the function $f(x) = \sqrt{x}$, whose derivative is $\frac{1}{2\sqrt{x}}$. For x close to 100 (as 99.8 is), we can use the linear approximation:

$$f(x) \approx f(100) + (x - 100)f'(100)$$

Since $f(100) = 10$ and $f'(100) = \frac{1}{2\sqrt{100}} = \frac{1}{20}$, it follows that

$$f(99.8) \approx 10 - 0.2 \cdot \frac{1}{20} = 9.9$$

For purposes of comparison, the actual value of $\sqrt{99.8}$ is around 9.989994995.

7. **(15 points)** *Amy is standing motionless* 50 *meters east of a north-south road with a radar gun, while Bob, who is* 120 *meters to the north, is driving south. The radar gun reports how quickly the* distance between Amy and Bob *is changing (which may not be Bob's actual speed).*

Our setup for this problem involves a right triangle between Bob's car, the place on the road parallel to Amy, and Amy's position well off the road. The east-west leg between Amy and the road is a constant length of 50, while the length of the north-south leg and the length of the hypotenuse are changing. We may denote these respectively by y and s, and note that, by the Pythagorean Theorem, they have the relationship $y^2 + 50^2 = s^2$. We also know that y is *currently* 120, and can compute in additon that s is currently

$$\sqrt{50^2 + y^2} = \sqrt{2500 + 14400} = \sqrt{16900} = 130$$

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(a) **(11 points)** *If Bob is driving south at* 30 *meters per second, what will the radar report as the rate of change of the distance between Bob and Amy?*

The information given to us here is that $\frac{dy}{dt} = -30$ (since y is decreasing at a rate of 30 meters per second), and that we wish to find $\frac{ds}{dt}$. Using the relationship described in the genral problem, we differentiate both sides with respect to t:

$$\begin{aligned} y^2 + 50^2 &= s^2 \\ \frac{d}{dt}\left(y^2 + 50^2\right) &= \frac{d}{dt}s^2 \\ \frac{dy}{dt}\frac{d}{dy}\left(y^2 + 50^2\right) &= \frac{ds}{dt}\frac{d}{ds}s^2 \\ \frac{dy}{dt}2y &= \frac{ds}{dt}2s \\ \frac{\frac{dy}{dt}y}{s} &= \frac{ds}{dt} \end{aligned}$$

and thus $\frac{ds}{dt} = \frac{-30 \cdot 120}{130} = \frac{-36}{13}$.

(b) **(4 points)** *Conversely, if the radar reported a change-rate of* 25 *meters per second, what would Bob's actual speed be?*

In this case we are given $\frac{ds}{dt} = \pm 25$, and asked to find $\frac{dy}{dt}$. We may start with the second-to last line of the derviation seen in the previous section:

$$\frac{dy}{dt}2y = \frac{ds}{dt}2s$$

and solve for $\frac{dy}{dt} = \frac{\frac{ds}{dt}s}{y} = \frac{\pm 25 \cdot 130}{120} = \pm\frac{325}{12}$.

8. **(10 points)** *Find an equation of the tangent line to the curve* $y = e^x(x^2 - 4x + 3)$ *at* $(0, 3)$.

Using the product rule,

$$\frac{dy}{dx} = e^x(x^2 - 4x + 3) + e^x(2x - 4)$$

and specifically when $x = 0$, $\frac{dy}{dx} = 1(3) + 1(-4) = -1$, so the desired line has slope -1. Since it passes through the point $(0, 3)$, it has point-slope form $y - 3 = -1(x - 0)$, or alternatively the slope-intercept form $y = -x + 3$.