1. **(12 points)** *Consider the function* $g(x) = \frac{e^x}{x-1}$.

   (a) **(5 points)** *Identify the zeroes, vertical asymptotes, and long-term behavior on both sides of this function. Clearly label which is which, and if any features are not present, say so.*

   We note that in order for $\frac{e^x}{x-1}$ to be zero, its numerator $e^x$ must be zero, but $e^x$ is always positive, so $g(x)$ has no zeroes.

   Vertical asymptotes occur when the denominator is zero; this occurs when $x = 1$.

   To calculate long-term behavior, we consider the behavior on the right, where a $\frac{\infty}{\infty}$ form calls for an invocation of L'Hôpital's rule:

   $$\lim_{x\to+\infty} \frac{e^x}{x-1} = \lim_{x\to+\infty} \frac{e^x}{1} = +\infty$$

   while on the left, $e^x \to 0$, so we have the form $\frac{0}{\infty}$, which is not indeterminate: $\lim_{x\to-\infty} g(x) = 0$.

   (b) **(5 points)** *Identify the critical points of this function, and indicate whether each is a local maximum, local minimum, or neither.*

   We calculate $g'(x)$:

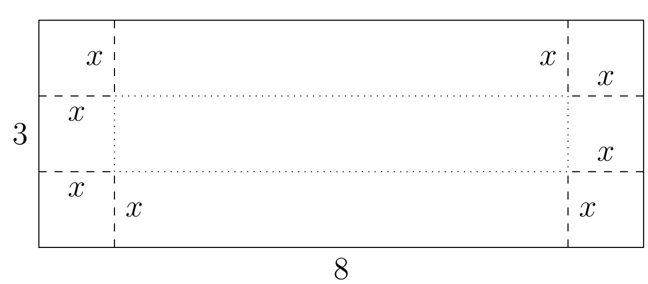
   $$\frac{d}{dx}\frac{e^x}{x-1} = \frac{(x-1)e^x - e^x}{(x-1)^2} = \frac{x-2}{(x-1)^2}e^x$$

   Now we find places where $g'(x)$ is zero or nonexistent; the numerator is zero when $x = 2$, and the denominator is zero, inducing nonexistence, when $x = 1$. The latter is neither a local minimum nor a maximum, since it's not even in the domain of $g$. The former, on the other hand, will be a minimum: $\frac{x-2}{(x-1)^2}e^x$ transitions from negative when $x < 2$ to positive when $x > 2$.

   (c) **(2 points)** *Which if any of the critical points identified above are global maxima or global minima? Show work or explain.*

   None of them are global maxima, since $g(x)$ gets arbitrarily large off to the right, and $x = 2$ will not in fact be a global minimum, either, since $\lim_{x\to-\infty} g(x) < g(2)$.

2. **(20 points)** *We have a rectangular sheet of cardboard which is 3 feet by 8 feet in dimensions. By cutting identical squares out of the corners and folding up the resulting sheet, we can build a box without a lid. What size on our corner cut maximizes the volume of the box so generated?*




Let the depth of the cut be $x$ (along the dashed lines depicted above). Then, folding this up gives a box whose base is of dimensions $(8-2x) \times (3-2x)$ and whose height is $x$. The total volume is thus $V(x) = x(8-2x)(3-2x) = 4x^3 - 22x^2 + 24x$; we seek to maximize this volume over the interval $(0, 1.5)$. $x$ must clearly be positive; it could not be 1.5 or more because then the base dimension $3 - 2x$ would be zero or negative. Now, we find the critical points of $V(x)$

by calculating $V'(x) = 12x^2 - 44x + 24 = 4(3x^2 - 11x + 6) = 4(3x-2)(x-3)$. This is zero when $x = \frac{2}{3}$ or $x = 3$; the latter is outside our domain, so our three candidates for area-maximizing behavior are the limiting behavior as $x$ approaches 0 from above, the limiting behavior as $x$ approaches 1.5 from below, and $x = \frac{2}{3}$. As might be expected, the first two are terrible:

$$\lim_{x \to 0^+} V(x) = V(0) = 0$$

$$\lim_{x \to 1.5^-} V(x) = V(1.5) = 0$$

so the third choice is the best (specifically, $V(\frac{2}{3}) = \frac{2}{3} \cdot \frac{20}{3} \cdot \frac{5}{3} = \frac{200}{27}$).

3. **(16 points)** *Calculate the following derivatives:*

(a) **(4 points)** *Find* $\frac{d}{dt} \int \frac{e^t}{t} dt$.

In general, $\frac{d}{dx} \int f(x)dx = f(x)$, so in this particular example, $\frac{d}{dt} \int \frac{e^t}{t} dt = \frac{e^t}{t}$.

(b) **(6 points)** *Find* $\frac{d}{dx} \arctan \frac{x^2-1}{x+2}$.

The expression being differentiated here is the arctangent of a quotient; we thus might reasonably expect to use the chain rule and the quotient rule. We might let $u = \frac{x^2-1}{x+2}$, and then invoke the chain rule:

$$\begin{aligned} \frac{d}{dx} \arctan \frac{x^2-1}{x+2} &= \frac{d}{dx} \arctan u \\ &= \frac{du}{dx} \frac{d}{du} \arctan u \\ &= \frac{d}{dx} \frac{x^2-1}{x+2} \frac{d}{du} \arctan u \\ &= \frac{(x+1)(2x) - (x^2-1)(1)}{(x+2)^2} \frac{1}{1+u^2} \\ &= \frac{(x+1)(2x) - (x^2-1)(1)}{(x+2)^2 (1 + \left(\frac{x^2-1}{x+2}\right)^2)} \end{aligned}$$

(c) **(6 points)** *Given* $g(s) = e^s \cot(s^2)$, *find* $g'(s)$.

The expression being differentiated here is a product, one of whose factors is a composition, so we might expect to use the product rule and then the chain rule. Defining $u = s^2$ pre-emptively, we start with the product rule, and introduce $u$ as necessary:

$$\begin{aligned} g'(s) &= \frac{d}{ds} \left( e^s \cot(s^2) \right) \\ &= e^s \cot(s^2) + e^s \frac{d}{ds} \cot(s^2) \\ &= e^s \cot(s^2) + e^s \frac{d}{ds} \cot(u) \\ &= e^s \cot(s^2) + e^s \frac{du}{ds} \frac{d}{du} \cot(u) \\ &= e^s \cot(s^2) + e^s \cdot 2s(-\csc^2 u) \\ &= e^s \cot(s^2) - 2se^s \cdot \csc^2(s^2) \end{aligned}$$

4. **(15 points)** *The* keratoid cusp *is a curve satisfying the equation* $y^2 = x^2y + x^5$.

(a) **(10 points)** *Find a formula for* $\frac{dy}{dx}$ *on this curve in terms of* $x$ *and* $y$.

We differentiate both sides with respect to $x$, and use the product and chain rule as appropriate:

$$\begin{aligned}
\frac{d}{dx}\left(y^2\right) &= \frac{d}{dx}\left(x^2y + x^5\right) \\
\frac{dy}{dx}\frac{d}{dy}y^2 &= 2xy + x^2\frac{dy}{dx} + 5x^4 \\
2y\frac{dy}{dx} &= 2xy + x^2\frac{dy}{dx} + 5x^4 \\
(2y - x^2)\frac{dy}{dx} &= 2xy + 5x^4 \\
\frac{dy}{dx} &= \frac{2xy + 5x^4}{2y - x^2}
\end{aligned}$$

(b) **(5 points)** *Find the equation of the tangent line to the keratoid cusp at the point* $(2, -4)$.

Using the result above at this point, we find that at $(2, -4)$, the slope of the tangent line is $\frac{dy}{dx} = \frac{2 \cdot 2(-4) + 5 \cdot 2^4}{2(-4) - 2^2} = \frac{64}{-12} = \frac{-16}{3}$.

The equation of the tangent line is thus, in point-slope form, $(y + 4) = \frac{-16}{3}(x - 2)$.

5. **(18 points)** *Evaluate the following integrals:*

(a) **(6 points)** $\int_{-1}^{2}(x^2 - 4)dx$.

Using the Fundamental Theorem of Calculus and the antiderivative power rule:

$$\int_{-1}^{2}(x^2 - 4)dx = \frac{x^3}{3} - 4x\Big]_{-1}^{2} = \frac{8}{3} - 8 - \left(\frac{-1}{3} + 4\right) = -9$$

(b) **(6 points)** $\int_0^2 \frac{2x}{(x^2+1)^2}dx$.

This is not a simple antiderivative, so we consider substitution possibilities. Since exponentiation is being applied to the complicated expression $x^2 + 1$, it is promising to consider $u = x^2 + 1$, which yields the pseudo-equation "$du = 2xdx$". Then, translating the above integral:

$$\int_0^2 \frac{2x}{(x^2 + 1)^2}dx = \int_{0^2+1}^{2^2+1} \frac{1}{u^2}du = \frac{-1}{u}\Big]_1^5 = \frac{-1}{5} - \left(\frac{-1}{1}\right) = \frac{4}{5}$$

(c) **(6 points)** $\int (t^5 - t)\sqrt{t^6 - 3t^2}dt$.

This is not a simple antiderivative, so we consider substitution possibilities. Since the square rootexponentiation is being applied to the complicated expression $t^6 - 3t^2$, it is promising to consider $u = t^6 - 3t^2$, which yields the pseudo-equation "$du = (6t^5 - 6t)dt$"; note that then $(t^5 - t)dt$ can be replaced with $\frac{1}{6}du$, so we translate the above integral:

$$\int (t^5 - t)\sqrt{t^6 - 3t^2}dt = \int \frac{1}{6}\sqrt{u}du = \frac{1}{6}\frac{u^{3/2}}{3/2} + C = \frac{(t^6 - 3t^2)^{3/2}}{9} + C$$

6. **(16 points)** *A sentry at Blackgate Prison has turned a spotlight on an escapee who is currently 0.3 miles to the north and 0.4 miles to the east of the prison. She notices that the escapee is traveling eastwards at three miles per hour.*

   (a) **(8 points)** *How quickly will she need to rotate the spotlight to keep it trained on the escapee?*

   Let the angle of the spotlight from true north be called $\theta$, and let the distance the fugitive is to the east of the prison be called $x$. Since the fugitive is a constant distance 0.3 to the north of the prison, and a distance $x$ easy of the prison, drawing a right triangle makes it clear that $\tan\theta = \frac{x}{0.3}$.

   We know that $x$ is currently 0.4, and that $\frac{dx}{dt} = 3$, so we may use related-rates techniques to determine $\frac{d\theta}{dt}$, differentiating each side of the above relationship with respect to $t$:

$$\frac{d}{dt}\tan\theta = \frac{d}{dt}\frac{x}{0.3}\frac{d\theta}{dt}\frac{d}{d\theta}\tan\theta \qquad = \frac{\frac{dx}{dt}}{0.3}\frac{d\theta}{dt}\sec^2\theta = \frac{\frac{dx}{dt}}{0.3}\frac{d\theta}{dt} \qquad = \frac{\frac{dx}{dt}\cos^2\theta}{0.3}$$

   and since $\theta$ is in a right triangle adjacent to a side of length 0.3 and with hypotenuse 0.5, we may specifically expand that to

$$\frac{d\theta}{dt} = \frac{\frac{dx}{dt}\cos^2\theta}{0.3} = \frac{3\left(\frac{0.3}{0.5}\right)^2}{0.3} = \frac{0.9}{0.25} = 3.6$$

   in radians per hour.

   (b) **(8 points)** *How quickly is the escapee's distance from the prison changing?*

$$\begin{aligned}
\frac{d}{dt}(0.3^2 + x^2) &= \frac{d}{dt}(s^2) \\
\frac{dx}{dt}\frac{d}{dx}(x^2) &= \frac{ds}{dt}\frac{d}{ds}(s^2) \\
\frac{dx}{dt}(2x) &= \frac{ds}{dt}(2s) \\
\frac{x\frac{dx}{dt}}{s} &= \frac{ds}{dt}
\end{aligned}$$

   And since $x$ and $s$ have current values of 0.4 and 0.5 respectively, $\frac{ds}{dt} = \frac{0.4\cdot 3}{0.5} = 2.4$

7. **(15 points)** *Determine the following limits.*

   (a) **(5 points)** *Evaluate* $\lim_{x\to 2}\frac{\sqrt[3]{x+6}-2}{x-2}$ *or demonstrate that it cannot be evaluated.*

   Direct evaluation gives $\frac{0}{0}$, so we might use L'Hôpital's rule:

$$\lim_{x\to 2}\frac{\sqrt[3]{x+6}-2}{x-2} = \lim_{x\to 2}\frac{\frac{1}{3}(x+6)^{-2/3}}{1} = \frac{\frac{1}{3}(8^{-2/3})}{1} = \frac{1}{12}$$

   (b) **(5 points)** *Evaluate* $\lim_{x\to\infty}\frac{x^3+x\ln x}{3x^2-5}$ *or demonstrate that it cannot be evaluated.*

   You might simply look at the dominant term in the numerator and denominator to find that $\lim_{x\to\infty}\frac{x^3+x\ln x}{3x^2-5} = \lim_{x\to\infty}\frac{x^3}{3x^2} = +\infty$; alternatively, L'Hôpital's rule can be used several times on indeterminate forms evaluating to $\frac{\infty}{\infty}$:

$$\lim_{x\to\infty}\frac{x^3+x\ln x}{3x^2-5} = \lim_{x\to\infty}\frac{3x^2+\ln x+1}{6x} = \lim_{x\to\infty}\frac{6x+\frac{1}{x}}{6} = \infty$$

(c) **(5 points)** *Using the difference quotient, find the derivative with respect to $x$ of $f(x) = 3x^2 - 4x + 2$. You may not use L'Hôpital's rule for this problem.*

$$\begin{aligned} f'(x) &= \lim_{x\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{x\to 0} \frac{(3(x+h)^2 - 4(x+h) + 2) - (3x^2 - 4x + 2))}{h} \\ &= \lim_{x\to 0} \frac{6xh + 3h^2 - 4h}{h} \\ &= \lim_{x\to 0} 6x + 3h - 4 = 6x - 4 \end{aligned}$$

8. **(15 points)** *Let $f(x) = 2x^3 + 3x^2 - 36x$.*

   (a) **(5 points)** *Where is $f(x)$ increasing? Where is it decreasing? Label which is which.*

   We consider the sign-changes of $f'(x) = 6x^2 + 6x - 36 = 6(x-2)(x+3)$. This is clearly zero at $x = 2$ and $x = -3$. Probing between and among the zeroes indicates that when $x < -3$, $f'(x)$ is positive, and when $-3 < x < 2$, $f'(x)$ is negative, and when $x > 2$, $f'(x)$ is positive again. Thus $f(x)$ is increasing on $(-\infty, -3)$ and $(2, \infty)$, and is decreasing on $(-3, 2)$.

   (b) **(3 points)** *What are the critical points of $f(x)$? Is each a local maximum, a local minimum, or neither?*

   Since $x = -3$ is a transition from increase to decrease, it is a local maximum, and since $x = 2$ is a transition from decrease to increase, it is a local minimum.

   (c) **(7 points)** *Determine where $f(x)$ is concave up and where it is concave down, and identify points of inflection.*

   We consider the sign-changes of $f''(x) = 12x + 6$. This is zero at $x = \frac{-1}{2}$, and probing to the left and right of this zero indicates that $f''(x) < 0$ when $x < \frac{-1}{2}$, and $f''(x) > 0$ when $x > \frac{-1}{2}$. Thus, $f(x)$ is concave down on $(-\infty, -\frac{1}{2})$ and concave up on $(-\frac{1}{2}, \infty)$.

9. **(8 points)** *Answer the following questions about the function $h(x) = \sqrt{25 - x^2}$.*

   (a) **(4 points)** *What is the domain of $h(x)$?*

   $h(x)$ is defined as long as the argument of the square root is non-negative, so wherever $25 - x^2 \geq 0$, whihc is when $x^2 \leq 25$, which is in the interval $[-5, 5]$.

   (b) **(4 points)** *Where does the derivative of $h(x)$ exist?*

   Using the chain rule, $g'(x) = \frac{-2x}{2\sqrt{25-x^2}}$. The argument of the square root must exist, limiting our options to the domain found in part (a), but in addition it must be nonzero, so we must additionally exclude values such that $25 - x^2 = 0$, which are $x = \pm 5$. Thus, our interval of differentiability is $(-5, 5)$.

10. **(15 points)** *Miranda has just taken a 40mg intravenous dose of G-23 paxilon hydrochlorate. In two hours the level of the drug in her system, which is subject to exponential decay, will have reduced to 25mg.*

(a) **(5 points)** *Construct a function modeling the quantity of the drug in her body after t hours.*

We know that this is an exponential decay system, so the function modeling the quantity of the drug is $f(t) = Ce^{kt}$ for some well-chosen $C$ and $k$. We also know, based on the situation described, that $f(0) = 40$ and $f(2) = 25$. Thus, using the first fact:

$$40 = f(0) = Ce^{0k}$$

so $C = 40$. Now, using the second fact:

$$\begin{aligned} 25 &= 40e^{2k} \\ \frac{5}{8} &= e^{2k} \\ \ln\frac{5}{8} &= 2k \\ \frac{\ln\frac{5}{8}}{2} &= k \end{aligned}$$

so $f(t) = 40e^{\frac{\ln(5/8)}{2}t}$.

(b) **(6 points)** *How quickly is the drug being eliminated after 2 hours?*

Using the chain rule, $f'(t) = 40 \cdot \frac{\ln\frac{5}{8}}{2} e^{\frac{\ln(5/8)}{2}t}$, so specifically

$$f'(2) = 20\ln\frac{5}{8}e^{\ln(5/8)}$$

This is approximately $-5.87$, so we may interpret this result to indicate that the drug is being eliminated at a rate of approximately 5.87 milligrams per hour.

(c) **(4 points)** *The experimental protocol requires that subjects be kept under careful observation until the level of the drug in their body is below 10mg. After how many hours can Miranda be released from observation?*

The question here is asking when $f(t) = 10$. This can be solved algebraically:

$$\begin{aligned} 40e^{\frac{\ln(5/8)}{2}k} &= 10 \\ e^{\frac{\ln(5/8)}{2}k} &= \frac{1}{4} \\ \frac{\ln\frac{5}{8}}{2}k &= \ln\frac{1}{4} \\ k &= \frac{2\ln\frac{1}{4}}{\ln\frac{5}{8}} \end{aligned}$$

This is approximately 5.9, so she can be released after nearly 6 hours.