This is the full proof of the Cantor-Schröder-Bernstein Theorem, a powerful result for developing relationships between different sets.

1 Statement of the Theorem, and a simple case

Theorem 1 (Dedekind 1887, Cantor 1895, Schröder 1896, Bernstein 1897, König 1906)**.** *For sets A and B, if $f : A \to B$ is an injective function and $g : B \to A$ is an injective function, then there is a bijective function $h : A \to B$.*

This turns out to be a very-nearly trivial result when A or B is finite; we'll prove that much simpler statement just to get a handle on the question being asked, although none of our methods will translate over.

Proposition 1. *For sets A and B, if either A or B is finite, and $f : A \to B$ is an injective function, and $g : B \to A$ is an injective function, then there is a bijective function $h : A \to B$.*

Proof. Recall a result from previously, which we will use a great deal in this proof: if B is finite and $f : A \to B$ is injective, then A is a finite set and $|A| \leq |B|$; in other words, an injective function with finite codomain has a domain which is finite and no larger than its codomain.

Note that if A is finite, then by the above result applied to $g : B \to A$, B is finite; likewise if B is finite, then by the above result applied to $f : A \to B$, A is finite, so from the premise that at least one of these two sets is finite, we may conclude that both are.

Now, by injectivity of f, it follows that $|A| \leq |B|$, and by injectivity of g, it follows that $|B| \leq |A|$. Thus $|A| = |B|$; let us denote the size of A by n, so that $|A| = |B| = n$. Then we may assign names to the n distinct elements of A and B as such: let $A = \{a_1, a_2, \ldots, a_n\}$ and $B = \{b_1, b_2, \ldots, b_n\}$, and now we can build a bijection by simply letting $h(a_i) = b_i$ for each i in $1, 2, \ldots, n$. This is clearly injective, since if $i \neq j$, then $a_i \neq a_j$ and $b_i \neq b_j$; likewise it is surjective, since for each $b_i \in B$, we have that $f(a_i) = b_i$. $\square$

Of course, for infinite sets, we have no such convenient notion of "size", and even the idea that we could label the elements of A and B in order turns out to be subtly flawed. So we must be much more careful in our actual proof. The following proof is based on König's argument; the previous proofs were more technical and used some unusual set-theoretical axioms.

2 The Zig-Zag Sequence and Partition of A

One key mechanism we will use is what might be called a "zig-zag sequence" built from the given injections $f : A \to B$ and $g : B \to A$. We will construct sequences which partition A and B, and on each part of this partition we'll build a bijection.

Let us consider some $a \in A$. In one direction, it is clear that we can trace the process of repeated applications of f and g to this element of A, and build a sequence of alternating elements from A and B in such a manner:

$$a, f(a), g(f(a)), f(g(f(a))), g(f(g(f(a)))), \ldots, (g \circ f)^n(a), (f \circ (g \circ f)^n)(a), (g \circ f)^{n+1}(a), \ldots$$

and we could continue this process indefinitely (although not every element of the sequence is guaranteed to be distinct).

We'd like to also step *backwards* from a; we shall see why this is useful as we analyze each individual sequence. Here we need to be careful: there is no good reason to believe there is any

b such that $g(b) = a$! But we will build inverse functions and be extremely careful about their application. $f : A \to B$ and $g : B \to A$, which may not be surjective, are not guaranteed to have inverses; however, by restricting their codomain, we can frame the same set of ordered pairs in a surjective *context*: let's define A' and B' to be the images of g and f respectively; then if we consider the reframing of the same functions as $f : A \to B'$ and $g : B \to A'$, their codomains are definitionally equal to their images so they are surjective, and we can assert the existence of inverses $f^{-1} : B' \to A$ and $g^{-1} : A' \to B$.

Now, with these inverses, we can attempt to build a backwards-stepping sequence from a:

$$a, g^{-1}(a), f^{-1}(g^{-1}(a)), g^{-1}(f^{-1}(g^{-1}(a))), \ldots, (f^{-1} \circ g^{-1})^n(a), (g^{-1} \circ (f^{-1} \circ g^{-1})^n)(a), (f^{-1} \circ g^{-1})^{n+1}(a), \ldots$$

But our blithe assertion that this process can be performed is predicated on some shaky ground; remember that f^{-1} and g^{-1} have domains A' and B', not A and B. This process could fail to be meaningful on the first step, if $a \notin A'$, or on the second step, if $g^{-1}(a) \notin B'$, or the third, if $f^{-1}(g^{-1}(a)) \notin A'$, and so forth.

We shall thus define a structure called the *zig-zag sequence* of a particular $a \in A$ to be one of the three following constructs.

- A sequence which has a "stopping point" in A:

$$(f^{-1} \circ g^{-1})^n(a), \left(g^{-1} \circ (f^{-1} \circ g^{-1})^{n-1}\right)(a), (f^{-1} \circ g^{-1})^{n-1}(a), \ldots,$$
$$(f^{-1} \circ g^{-1})(a), g^{-1}(a), a, f(a), (g \circ f)(a), (f \circ g \circ f)(a), (g \circ f \circ g \circ f)(a), \ldots$$

  where $(f^{-1} \circ g^{-1})^n(a) \notin A'$,

- a sequence which has a "stopping point" in B:

$$\left(g^{-1} \circ (f^{-1} \circ g^{-1})^n\right)(a), (f^{-1} \circ g^{-1})^n(a), \left(g^{-1} \circ (f^{-1} \circ g^{-1})^{n-1}\right)(a), (f^{-1} \circ g^{-1})^{n-1}(a), \ldots,$$
$$(f^{-1} \circ g^{-1})(a), g^{-1}(a), a, f(a), (g \circ f)(a), (f \circ g \circ f)(a), (g \circ f \circ g \circ f)(a), \ldots$$

  where $\left(g^{-1} \circ (f^{-1} \circ g^{-1})^n\right)(a) \notin B'$,

- or a sequence which does not stop in its backwards propogation:

$$\ldots, (g^{-1} \circ f^{-1} \circ g^{-1})(a), (f^{-1} \circ g^{-1})(a), g^{-1}(a), a, f(a), (g \circ f)(a), (f \circ g \circ f)(a), (g \circ f \circ g \circ f)(a), \ldots$$

These three types of sequences have traditionally been handled separately; in addition, the third case is often considered with regard to two distinct possibilities, where all terms of the sequence are distinct or where they are not all distncit. As it turns out, we actually only need to consider *two* distinct cases when we finally get to analyzing individual zig-zag sequences.

First, however, we will have to carve A and B up by using these sequences. We will show that it is possible to do so by defining a relation on shared-sequence-membership, and that this relation is actually an equivalence relation.

Let us define a relation R on A as such: for $x, y \in A$, $x \; R \; y$ if and only if y appears in the zig-zag sequence of x: that is to say, if either $y = x$ or there is some positive integer n such that either $(g \circ f)^n(x) = y$ or $(f^{-1} \circ g^{-1})^n(x) = y$.

Lemma 1. *R is an equivalence relation.*

Proof. First we shall demonstrate reflexivity: for every $x \in A$, the defined construction of x's zig-zag sequence always includes x between the forward- and backward-propagations, so clearly x lies in the zig-zag sequence of x and $x\ R\ x$.

Now we demonstrate symmetry: let us assume the symmetric premise of specific $x, y \in A$ such that $x\ R\ y$, so either $y = x$ (in which case $y\ R\ x$ trivially), or $y = (g \circ f)^n(x)$, or $y = (f^{-1} \circ g^{-1})^n(x)$. If $y = (g \circ f)^n(x)$, we can perform inverses step by step:

$$\begin{aligned} y &= (g \circ f)^n(x) \\ g^{-1}(y) &= \left(g^{-1} \circ g \circ f \circ (g \circ f)^{n-1}\right)(x) \\ g^{-1}(y) &= \left(f \circ (g \circ f)^{n-1}\right)(x) \\ (f^{-1} \circ g^{-1})(y) &= \left(f^{-1} \circ f \circ (g \circ f)^{n-1}\right)(x) \\ (f^{-1} \circ g^{-1})(y) &= \circ (g \circ f)^{n-1}(x) \end{aligned}$$

and continue inductively until we determine that

$$(f^{-1} \circ g^{-1})^n(y) = x$$

which is a criterion for showing that $y\ x$.

Likewise, when $y = (f^{-1} \circ g^{-1})^n(x)$, we can disentangle x by repeated applications of f and g:

$$\begin{aligned} y &= (f^{-1} \circ g^{-1})^n(x) \\ f(y) &= \left(f \circ f^{-1} \circ g^{-1} \circ (f^{-1} \circ g^{-1})^{n-1}\right)(x) \\ f(y) &= \left(g^{-1} \circ (f^{-1} \circ g^{-1})^{n-1}\right)(x) \\ (g \circ f)(y) &= \left(g \circ g^{-1} \circ (f^{-1} \circ g^{-1})^{n-1}\right)(x) \\ (g \circ f)(y) &= \circ (f^{-1} \circ g^{-1})^{n-1}(x) \end{aligned}$$

so again using induction we could show that

$$(g \circ f)^n(y) = x$$

so that $y\ R\ x$.

Finally, we must show transitivity, which ends up being surprisingly simple: there are six cases but they are largely identical: we compose two functions of the forms $(g \circ f)^n$ or $(f^{-1} \circ g^{-1})^n$, and it is easy to show we get back a function of the same form. □

Because being-in-the-same-sequence is an equivalence relation, the equivalence classes (i.e., the sets of elements of A which lie in a single sequence) partition A. Thus, each element of A lies in exactly one zig-zag sequence. It is easy to show that every element of B also lies in exactly one zig-zag sequence: if $b \in B$, then b lies in the sequence determined by $g(b)$; if b lay in multiple sequences those multiple sequences would both contain $g(b)$.

This decomposition gives us a framework for our proof: if we can build a bijection between the elements of A and B within the terms of a single sequence, and then repeat that procedure for every sequence, then we'll have constructed a bijection between A and B. Henceforth, we will look not at A and B as a whole, but at the sets S_A and S_B consisting of the elements from A and B of a specific zig-zag sequence.

On each sequence, we will find that a *restriction* of the function f or g suffices; let us define a restriction explicitly:

Definition 1. For a function $f : A \to B$ and $S \subseteq A$, the *restriction of f to S*, written f_S, is a function from S to B given by $f_S(a) = f(a)$.

There are four essential "shapes" the zig-zag sequences can have, which have traditionally been handled separately: there are sequences where the inverse functions remain applicable at every backwards step and which never repeat themselves, there are sequences where the inverse functions remain applicable at every backwards step and which do repeat themselves, there are sequences which eventually reach some element of A on which g^{-1} is not defined, and there are sequences which eventually reach some element of B on which f^{-1} is not defined. For simplicity, we can actually fold the first three prospects into a single case.

3 Zig-Zag sequences without B-blockage

We shall consider in this section a specific type of zig-zag sequence: we shall look at the case where $S_B \subseteq B'$, which is to say, f^{-1} is defined on each element of S_B. We shall show the following significant proposition for such zig-zag sequences:

Lemma 2. *If the sets $S_A \subseteq A$ and $S_B \subseteq B$ are the terms of a zig-zag sequence, and if furthermore $S_B \subseteq B'$, then the restriction f_{S_A} is a bijection from S_A to S_B.*

Proof. Note that as traditionally defined, f_{S_A} has domain S_A and codomain B, and is furthermore injective, since it is a restriction of a function which was injective. However, we want to show that, considered with codomain S_B, it is also surjective; thus, we want to show that the image of f_{S_A} is actually S_B; to do so, we must first show that for all $a \in S_A$, it is the case that $f(a) \in S_B$; in addition, we must show that for all $b \in S_B$, there is some $a \in S_A$ such that $f(a) = b$.

In order to deal with the first assertion, let us consider an arbitrary $a \in S_A$. Since we showed that every element of a particular zig-zag sequence's S_A generates the exact same sequence (since they lie in the same equivalence class under the members-of-the-same-sequence relation explored above), we may assert that the zig-zag sequence of which S_A and S_B are the terms was in fact generated by a, so one forward step from this generator yields $f(a)$ as a term of the sequence, so $f(a) \in S_B$.

On the other hand, if we take an arbitrary $b \in S_B$, since $S_B \subseteq B'$, $b \in B'$, so since b is in the domain of f^{-1}, we may assert that $f^{-1}(b) \in A$. The zig-zag sequence generated by $f^{-1}(b)$ has one forward-step from its generator the term $f(f^{-1}(b))$, which is simply b. Since b lies in exactly one zig-zag sequence, it follows that $f^{-1}(b)$ generates the exact same zig-zag sequence as the one whose terms were distributed into S_A and S_B. Thus $f^{-1}(b) \in S_A$, and $f(f^{-1}(b)) = b$, so we have shown the existence of $a \in S_A$, specifically $a = f^{-1}(b)$, such that $f(a) = b$..

Thus f_{S_A} is surjective, when contextualized with codomain S_B, and as previously seen it is injective, so $f_{S_A} : S_A \to S_B$ is a bijection. □

Thus, among the parts of A and B described by zig-zag sequences which do not have terms in $B - B'$, we may restrict f to achieve bijections on these parts.

4 Zig-Zag sequences with B-blockage

Here we consider the case not addressed in the previous section, where $S_B \not\subseteq B'$; that is to say, sequences in which not every element of S_B lies in the domain of f^{-1}. Note that in this case the argument from the previous section will not work: elements of $B - B'$ aren't in the image of f, so a restriction of f won't be a surjection into S_B.

Now, as it turns out, the entirety of S_A must lie within A', which will allow us to craft a bijection not with f, but with g.

Proposition 2. *If the sets $S_A \subseteq A$ and $S_B \subseteq B$ are the terms of a zig-zag sequence and $S_B \nsubseteq B'$, then it is the case that $S_A \subseteq A'$, and furthermore $|S_B - B'| = 1$.*

Proof. Since $S_B \nsubseteq B'$, there is at least one element of S_B which is not an element of B'. Let $b \in (S_B - B')$. Since $b \in S_B$, it follows by the process defining the zig-zag sequence that $g(b) \in S_A$, so the zig-zag sequence whose terms provide the elements of S_A and S_B is equivalent to the sequence generated by $g(b)$. Identifying our sequence by this generator allows us to explicitly list both the forward-propogated and backwards-propagated elements of the sequence:

$$b, g(b), f(g(b)), g(f(g(b))), f(g(f(g(b)))), \ldots$$

We can be assured that the back-propogation is exactly one step, since $g(b)$ is definitionally in the image of g, so $g(b) \in A'$, but by our premise $b \notin B'$, so no back propagation further than b is possible.

It is now easy to show that every element of S_A is in A'. The terms this sequence contributes to S_A are $g(b)$, $g(f(g(b)))$, $g(f(g(f(g(b)))))$, and so forth; thus they are terms of the form $g\big[(f \circ g)^n(b)\big]$; since this term is of the form $g(b')$ for some $b' \in B$ (specifically given by $b' = (f \circ g)^n(b)$), it is clear that each $g\big[(f \circ g)^n(b)\big]$ is in the image of g, and thus lies in A'; since every element of S_A is in A', it follows that $S_A \subseteq A'$.

Similarly, the terms this sequence contributes to S_B are b, $f(g(b))$, $f(g(f(g(b))))$, and so forth. Every term of this list except for the first one is of the form $f(a)$ for some $a \in A$ particularly, of the form $a = g \circ (f \circ g)^n(b))$, so every element of S_B except for b lies in the image of f, so there is only one element of S_B which is not an element of B'. □

Since we know $S_A \subseteq A'$, we can craft a bijection using g^{-1} following almost exactly the logic in the previous section.

Lemma 3. *If the sets $S_A \subseteq A$ and $S_B \subseteq B$ are the terms of a zig-zag sequence and $S_B \nsubseteq B'$, then the restriction $g^{-1}_{S_A}$ is a bijection from S_A to S_B.*

Proof. By the previous proposition, given $S_B \nsubseteq B'$, we know $S_A \subseteq A'$; since g^{-1} has domain A', we are thus justified in defining a restriction onto its subset S_A.

As traditionally defined, $g^{-1}_{S_A}$ has domain S_A and codomain B. It is furthermore injective, since g^{-1} was injective and it is a restriction of a function which was injective. However, we want to show that, considered with codomain S_B, it is also surjective; thus, we want to show that the image of $g^{-1}_{S_A}$ is actually S_B; to do so, we must first show that for all $a \in A$, it is the case that $g^{-1}(a) \in S_B$; in addition, we must show that for all $b \in S_B$, there is some $a \in S_A$ such that $g^{-1}(a) = b$.

In order to deal with the first assertion, let us consider an arbitrary $a \in S_A$. Since we showed that every element of a particular zig-zag sequence's S_A generates the exact same sequence (since they lie in the same equivalence class under the members-of-the-same-sequence relation explored above), we may assert that the zig-zag sequence of which S_A and S_B are the terms was in fact generated by a. Since $S_A \subseteq A'$, it follows that $a \in A'$, so $g^{-1}(a)$ is defined, and one backwards step from this generator yields $g^{-1}(a)$ as a term of the sequence, so $g^{-1}(a) \in S_B$.

On the other hand, if we take an arbitrary $b \in S_B$, we may assert that $g(b) \in S_A$ by forward-propagation through the zig-zag sequence. Then note that since $g(b) \in A'$, $g(b)$ is in the domain of g^{-1}, so $g^{-1}(g(b)) = b$; thus $a = g(b)$ is an element of S_A such that $g^{-1}(a) = b$.

Thus $g^{-1}_{S_A}$ is surjective, when contextualized with codomain S_B, and as previously seen it is injective, so $f_{S_A} : S_A \to S_B$ is a bijection. □

5 The Explicit Construction, Recapitulated

In the end, the actual construction here is tedious (and difficult to actually perform, since certain steps involve an infinite number of steps, since both the number and lengths of zig-zag sequences may be infinite. However, the procedure used to construct our bijection $h : A \to B$ is really very simple.

Provided with sets A, B, and injections $f : A \to B$ and $g : B \to A$, we produce an assignment rule for each $a_0 \in A$ as such: produce by back-propagation a sequence $b_1 = g^{-1}(a_0)$, $a_1 = f^{-1}(b_1)$, $b_2 = g^{-1}(a_1)$, $a_2 = f^{-1}(b_2)$, and so forth. If at any point in this procedure, some $b_i \notin B'$, then let $h(a_0) = g^{-1}(a_0)$. Otherwise, let $h(a_0) = f(a_0)$.

This (possibly infinite-time to determine) criterion will determine h on every point of the codomain, and the justifications in the previous two sections serve to show that H is a bijection.