1. **(13 points)** Let $f(x) = 2x^2 - 5x + 2$.

   (a) **(9 points)** *Using the difference quotient, determine the formula for $f'(x)$.*

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{[2(x+h)^2 - 5(x+h) + 2] - (2x^2 - 5x + 2)}{h} \\
&= \lim_{h\to 0} \frac{(2x^2 + 4xh + 2h^2 - 5x - 5h + 2) - (2x^2 - 5x + 2)}{h} \\
&= \lim_{h\to 0} \frac{4xh + 2h^2 - 5h}{h} \\
&= \lim_{h\to 0} 4x + 2h - 5 \\
&= 4x - 5
\end{aligned}
$$

   (b) **(4 points)** *Find the equation of the tangent line to $f(x)$ at the point $(2, 0)$.*

   We know this line must pass through $(2, 0)$, with slope $f'(2) = 4 \cdot 2 - 5 = 3$. Using point-slope form, we get

$$y - 0 = 3(x - 2)$$

   which can also be expressed in slope-intercept form as $y = 3x - 6$.

2. **(9 points)** *Let* $f(x) = \begin{cases} x^3 - 6x & \text{if } x < 3 \\ ax + 2 & \text{if } 3 \le x < 9 \\ \sqrt{x} + b & \text{if } x \ge 9 \end{cases}$.

   What choices of a and b will make this function continuous everywhere?

   Each of the individual parts of this function can be easily observed to be continuous on its domain, so problems can only arise at the junction points $x = 3$ and $x = 9$. To guarantee continuity at these points, we need to make sure that the left and right limits coincide, as such at $x = 3$:

$$
\begin{aligned}
\lim_{x\to 3-} f(x) &= \lim_{x\to 3+} f(x) \\
\lim_{x\to 3-} x^3 - 6x &= \lim_{x\to 3+} ax + 2 \\
9 &= 3a + 2 \\
\frac{7}{3} &= a
\end{aligned}
$$

   And likewise for $x = 9$:

$$
\begin{aligned}
\lim_{x\to 9-} f(x) &= \lim_{x\to 9+} f(x) \\
\lim_{x\to 9-} ax + 2 &= \lim_{x\to 9+} \sqrt{x} + b \\
\frac{7}{3} \cdot 9 + 2 &= \sqrt{9} + b \\
20 &= b
\end{aligned}
$$

   So our solution is to choose $a = \frac{7}{3}$ and $b = 20$.

3. **(10 points)** Given the function $f(x) = \frac{4x^5-2x+1}{x^2+3x-4}$, answer the following questions preparatory to sketching the functions.

   (a) **(3 points)** *What is the domain of the function?*

   This function is evaluatable as long as the denominator of the fraction is nonzero; thus, the function's domain is the set of values where $x^2+3x-4 \neq 0$; in other words, when $x \neq -4, 1$. In interval form, this would be $(-\infty, -4) \cup (-4, 1) \cup (1, \infty)$.

   (b) **(7 points)** *Describe, either in words or symbolically, the long-term behavior of the function in each direction.*

   For very large positive, or large-magnitude negative, values of x, the comparative magnitudes of $4x^5$, $2x$ and 1 will be such as to make the latter two a negligible contribution to the numerator; likewise, in the denominator, x^2+3x-4 will resemble x^2 if x is of extraordinary magnitude.

   Thus, for very large-magnitude values of x, we see that $f(x) \approx \frac{4x^5}{x^2} = 4x^3$, which is a cubic function with positive coefficient. Thus, our reasonable understanding is that for x of very large magnitude, whether positive or negative, $f(x)$ will be of large magnitude and of the same sign as x. Alternatively, one might write symbolically that

$$\lim_{x\to+\infty} f(x) = +\infty, \quad \lim_{x\to-\infty} f(x) = -\infty$$

4. **(20 points)** *Evaluate the following limits; when a limit can not be evaluated, explicitly say so.*

   (a) **(4 points)** $\lim\limits_{u\to1} \frac{2u^2-u+1}{u^2-5u+4}$.

   Direct substitution into this limit gives $\frac{2}{0}$, which, since it has a nonzero numerator and a zero denominator, demonstrates that the limit does not exist (in an asymptotic way).

   (b) **(4 points)** $\lim\limits_{t\to\infty} \frac{t^3-7t^2+t}{2-4t^3}$.

   In the long term this function is dominated by its highest-degree terms in the numerator and denominator, so

$$\lim_{t\to\infty} \frac{t^3-7t^2+t}{2-4t^3} = \lim_{t\to\infty} \frac{t^3}{-4t^3} = \frac{-1}{4}$$

   Alternatively, one may formally divide by the highest degree appearing in the denominator, although doing so is a bit more involved:

$$\lim_{t\to\infty} \frac{t^3-7t^2+t}{2-4t^3} = \lim_{t\to\infty} \frac{\frac{t^3-7t^2+t}{t^3}}{\frac{2-4t^3}{t^3}} = \lim_{t\to\infty} \frac{1-\frac{7}{t}+\frac{1}{t^2}}{\frac{2}{t^3}-4} = \frac{1-0+0}{0-4} = \frac{-1}{4}$$

   (c) **(4 points)** $\lim\limits_{x\to3} \frac{x-3}{x^2-6x+9}$.

   Direct substitution yields the incalculable form $\frac{0}{0}$, and we shall thus try to factor out $(x-3)$ from the numerator and denominator. Since we look near, but not at, $x=3$, we can justify the straightforward cancellation arising therefrom:

$$\lim_{x\to3} \frac{x-3}{x^2-6x+9} = \lim_{x\to3} \frac{x-3}{(x-3)(x-3)} = \lim_{x\to3} \frac{1}{x-3}.$$

   Direct substitution into this new limit yields the result $\frac{1}{0}$, which, since it has a nonzero numerator and a zero denominator, demonstrates that the limit does not exist (in an asymptotic way).

(d) **(4 points)** $\lim_{s\to-\infty} \frac{s^4-3s^2+2s}{3s^7+s-1}$.

In the long term this function is dominated by its highest-degree terms in the numerator and denominator, so

$$\lim_{s\to-\infty} \frac{s^4-3s^2+2s}{3s^7+s-1} = \lim_{s\to-\infty} \frac{s^4}{3s^7} = \lim_{s\to-\infty} \frac{1}{3s^3} = 0.$$

Alternatively, one may formally divide by the highest degree appearing in the denominator, although doing so is a bit more involved:

$$\lim_{s\to-\infty} \frac{s^4-3s^2+2s}{3s^7+s-1} = \lim_{s\to-\infty} \frac{\frac{s^4-3s^2+2s}{s^7}}{\frac{3s^7+s-1}{s^7}} = \lim_{s\to-\infty} \frac{\frac{1}{s^3}-\frac{3}{s^5}+\frac{2}{s^6}}{3+\frac{1}{s^6}-\frac{1}{s^7}} = \frac{0-0+0}{3+0-0} = 0.$$

(e) **(4 points)** $\lim_{r\to 0} \frac{3e^r-r^2+1}{r+2\cos r}$.

Since the numerator and denominator are continuous throughout their domains, we may use straightforward substitution as long as the denominator is nonzero, which indeed it is:

$$\lim_{r\to 0} \frac{3e^r-r^2+1}{r+2\cos r} = \frac{3e^0-0^2+1}{0+2\cos 0} = \frac{4}{2} = 2.$$

5. **(16 points)** *Calculate the following derivatives, using any method you wish.*

(a) **(4 points)** *For $y = 6e^t - \frac{5}{t^3} + 2t - 23$, calculate its second derivative $\frac{d^2y}{dt^2}$.*

We may start by rewriting it so that each term is a power function or exponential function, and then use the appropriate differentiation rules twice:

$$\begin{aligned} y &= 6e^t - 5t^{-3} + 2t - 23 \\ \frac{dy}{dt} &= 6e^t + 15t^{-4} + 2 \\ \frac{d^2y}{dt^2} &= 6e^t - 60t^{-4} \end{aligned}$$

(b) **(4 points)** *Given that $f(x) = \frac{e^x-2x^3}{x^2-1}$, find $f'(x)$.*

This is a quotient, so we use the quotient rule:

$$\begin{aligned} f'(x) &= \frac{(x^2-1)\frac{d}{dx}(e^x-2x^3) - (e^x-2x^3)\frac{d}{dx}(x^2-1)}{(x^2-1)^2} \\ &= \frac{(x^2-1)(e^x-6x^2) - (e^x-2x^3)(2x)}{(x^2-1)^2} \end{aligned}$$

which need not be simplified further.

(c) **(4 points)** *Calculate $\frac{d}{dz}\left(7z^8 + 9\sqrt[4]{z} - 3e^z + \frac{18}{z^2}\right)$.*

If we rewrite this expression as $\frac{d}{dz}(7z^8 + 9z^{1/4} - 3e^z + 18z^{-2}$, it is easy to use the power rule or exponential rule on each term in turn to get $56z^7 + \frac{9}{4}z^{-3/4} - 3e^z - 36z^{-3}$.

(d) **(4 points)** *Calculate* $\frac{d}{du}\big((e^u+3)(4u^2-\sqrt{u})\big)$.
The expression being differentiated is a product of two expressions, so it is best to use the product rule.

$$\begin{aligned}\frac{d}{du}\big((e^u+3)(4u^2-\sqrt{u})\big) &= \big(\frac{d}{du}(e^u+3)\big)(4u^2-\sqrt{u}) + (e^u+3)\frac{d}{du}(4u^2-\sqrt{u})\\ &= \big(e^u\big)(4u^2-\sqrt{u}) + (e^u+3)\frac{d}{du}(4u^2-u^{1/2})\\ &= e^u(4u^2-\sqrt{u}) + (e^u+3)(8u-\frac{1}{2}u^{-1/2})\end{aligned}$$

which need not be simplified further.

6. **(9 points)** *Let* $f(x) = 6-2x$.

(a) **(1 point)** *Find* $\lim_{x\to-2} f(x)$.
Since $f(x)$ is a polynomial and thus continuous everywhere, direct substitution makes calculating this limit trivial: $\lim_{x\to-2} f(x) = f(-2) = 6-2(-2) = 10$.

(b) **(8 points)** *Using epsilon-delta methods, justify your result above.*
Given a value of ϵ, we constrain $f(x)$ to be within ϵ of 10, and attempt to derive a sufficient bound on δ therefrom:

$$\begin{aligned}|6-2x-10| &< \epsilon\\ |-2x-4| &< \epsilon\, |x+2| < \frac{\epsilon}{2}\end{aligned}$$

So, since it is sufficient to require x within $\frac{\epsilon}{2}$ of -2, we may establish δ to be $\frac{\epsilon}{2}$.

7. **(16 points)** *Determine the domains of the following functions:*

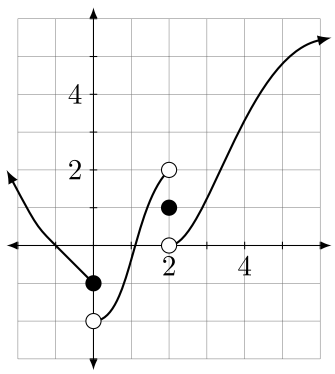
(a) **(4 points)** $f(u) = 7u^4 - \sqrt[3]{4u}$.
Since each individual element of this function has no obstructions to its calculation, this function has a domain consisting of all real numbers, or $(-\infty,\infty)$.

(b) **(4 points)** $g(x) = \frac{\sqrt{3x-4}}{x^2+9}$.
This function is evaluatable as long as the argument of the square root is non-negative and the denominator of the fraction is nonzero; thus, the function's domain is the set of values where $3x+4 \geq 0$ and $x^2+9 \neq 0$. The latter condition, however, is always true, since $x^{+}9$ must be at least 9 for all real x. Thus our only actual restriction is that $x \geq \frac{-4}{3}$, which could be written in interval notation as $[\frac{-4}{3},\infty)$.

(c) **(4 points)** $h(\theta) = \sin(3\theta) - 4\ln(6-\theta)$.
This function is evaluatable as long as the argument of the logarithm is positive (since the sine is defined everywhere); thus, the function's domain is the set of values where $6-\theta > 0$; in other words, when $\theta < 6$. In interval form, this would be $(-\infty, 6)$.

(d) **(4 points)** $p(r) = \frac{\sqrt{r^2-16}}{2r+12}$.
This function is evaluatable as long as the argument of the square root is non-negative and the denominator of the fraction is nonzero; thus, the function's domain is the set of values where $r^2 - 16 \geq 0$ and $2r + 12 \neq 0$; in other words, when $|r| \geq 4$ and $r \neq -6$, which could be further subdivided into requiring that either $r \leq -4$ but $r \neq -6$ or $r \geq 4$. In interval form, this would be $(-\infty, -6) \cup (-6, -4] \cup [4, \infty)$.

8. **(7 points)** *For the plot of $f(x)$ shown below, indicate whether or not each of the following quantities can be evaluated. If they can be evaluated, compute their values. If they cannot be evaluated, explicitly say so. You need not show work.*



$\lim_{x\to-1} f(x) = 0$, since the curve passes through $(-1, 0)$ without discontinuity.

$\lim_{x\to0^+} f(x) = -2$, since immediately to the right of 0, the graph is very close to the height $y = -2$.

$f(0) = -1$; note the solid dot at $(0, -1)$.

$\lim_{x\to0} f(x)$ does not exist, since the left and right limits at $x = 0$ do not coincide.

$f(2) = 1$; note the solid dot at $(2, 1)$.

$\lim_{x\to2^-} f(x) = 2$, since immediately to the left of 2, the graph is very close to the height $y = 2$.

$\lim_{x\to2^+} f(x) = 0$, since immediately to the right of 2, the graph is very close to the height $y = 0$.

9. **(5 point bonus)** *Determine a general rule by which the arbitrarily high-order derivative $\frac{d^n}{dx^n}\frac{x}{e^x}$ can be calculated (e.g. a specific formula in terms of n).*

We can simplify the first few derivatives and look for patterns:

$$\frac{d}{dx}\frac{x}{e^x} = \frac{e^x \cdot 1 - xe^x}{e^{2x}} = \frac{1-x}{e^x}$$
$$\frac{d^2}{dx^2}\frac{x}{e^x} = \frac{d}{dx}\frac{1-x}{e^x} = \frac{e^x(-1) - (1-x)e^x}{e^{2x}} = \frac{x-2}{e^x}$$
$$\frac{d^3}{dx^3}\frac{x}{e^x} = \frac{d}{dx}\frac{x-2}{e^x} = \frac{e^x \cdot 1 - (x-2)e^x}{e^{2x}} = \frac{3-x}{e^x}$$

And the pattern we see emerging is that

$$\frac{d^n}{dx^n}\frac{x}{e^x} = \frac{(-1)^n(x-n)}{e^x}$$