

1. **(20 points)** Answer the following derivative-related questions.

(a) **(8 points)** Calculate  $\frac{d}{dx} \sin(\ln(\arctan x))$ .

Initial analysis suggests that we will be using the chain rule twice (since this expression is the sine of a complicated expression, which is itself the logarithm of a complicated expression). We shall define  $v = \arctan x$  and  $u = \ln v$ , and then:

$$\begin{aligned} \frac{d}{dx} \sin(\ln(\arctan x)) &= \frac{d}{dx} \sin u \\ &= \frac{du}{dx} \frac{d}{du} \sin u \\ &= \left( \frac{d}{dx} \ln v \right) \cos u \\ &= \left( \frac{dv}{dx} \frac{d}{dv} \ln v \right) \cos u \\ &= \left( \frac{d}{dx} \arctan x \right) \cdot \frac{1}{v} \cdot \cos u \\ &= \frac{1}{1+x^2} \cdot \frac{1}{v} \cdot \cos u = \frac{\cos(\ln(\arctan x))}{(1+x^2) \arctan x} \end{aligned}$$

(b) **(6 points)** If  $f(q) = \frac{\ln q}{q^4}$ , find  $f'(q)$ .

Since we are taking a derivative of a quotient, we may approach this question with the quotient rule:

$$f'(q) = \frac{q^4 \left( \frac{d}{dq} \ln q \right) - \ln q \frac{d}{dq} q^4}{(q^4)^2} = \frac{q^4 \cdot \frac{1}{q} - \ln q \cdot 4q^3}{(q^4)^2} = \frac{1 - 4 \ln q}{q^5}$$

The last step is a simplification and is not necessary when solving this problem.

(c) **(6 points)** If  $y = \arcsin(3x) \tan x$ , find  $\frac{dy}{dx}$ .

This is a derivative of a product, which will require the product rule; there is also a chain rule within one of the factors, but that can be done implicitly rather than by a direct invocation of the chain rule.

$$\frac{dy}{dx} = \left( \frac{d}{dx} \arcsin(3x) \right) \tan x + \arcsin 3x \frac{d}{dx} \tan x = 3 \frac{1}{\sqrt{1-(3x)^2}} \tan x + \arcsin 3x \sec^2 x$$

2. **(10 points)** If  $y = \csc \frac{t^3-3t}{e^t+\sin t}$ , find  $\frac{dy}{dt}$ .

This expression will require the use of the chain rule (since on its outermost layer it is a cosecant of a complicated expression) and then the quotient rule (since that complicated expression is itself a quotient). We let  $u = \frac{t^3-3t}{e^t+\sin t}$  and proceed to invoke these two rules as

necessary:

$$\begin{aligned}
 \frac{dy}{dt} &= \frac{d}{dt} \csc u \\
 &= \frac{du}{dt} \frac{d}{du} \csc u \\
 &= \left( \frac{d}{dt} \frac{t^3 - 3t}{e^t + \sin t} \right) (-\csc u \cot u) \\
 &= \frac{(e^t + \sin t) \frac{d}{dt} (t^3 - 3t) - (t^3 - 3t) \frac{d}{dt} (e^t + \sin t)}{(e^t + \sin t)^2} (-\csc u \cot u) \\
 &= \frac{(e^t + \sin t)(3t^2 - 3) - (t^3 - 3t)(e^t + \cos t)}{(e^t + \sin t)^2} \left( -\csc \frac{t^3 - 3t}{e^t + \sin t} \cot \frac{t^3 - 3t}{e^t + \sin t} \right)
 \end{aligned}$$

3. **(10 points)** Calculate  $\frac{d}{ds}(\cos(s) \tan(s^2 - 3s))$ .

This expression will require the use of the product rule (since on its outermost layer it is a product of two factors) and then the chain rule (since one of those factors is the tangent of a complicated expression). We let  $u = s^2 - 3s$  and proceed to invoke these two rules as necessary:

$$\begin{aligned}
 \frac{d}{ds}(\cos(s) \tan(s^2 - 3s)) &= \frac{d}{ds}(\cos(s) \tan(u)) \\
 &= \left( \frac{d}{ds} \cos(s) \right) \tan(u) + \cos(s) \left( \frac{d}{ds} \tan(u) \right) \\
 &= -\sin(s) \tan(u) + \cos(s) \left( \frac{du}{ds} \frac{d}{du} \tan(u) \right) \\
 &= -\sin(s) \tan(u) + \cos(s) \left( \frac{d}{ds} s^2 - 3s \right) \sec^2(u) \\
 &= -\sin(s) \tan(u) + \cos(s)(2s - 3) \sec^2(u) \\
 &= -\sin(s) \tan(s^2 - 3s) + \cos(s)(2s - 3) \sec^2(s^2 - 3s)
 \end{aligned}$$

4. **(12 points)** A pure 150-gram sample of the radioactive material Cobalt-Thorium-G is taken to a shielded laboratory for testing. 6 days later it is found that, due to radioactive decay, only 125 grams of CoTh-G remain.

- (a) **(5 points)** Produce a function  $f(t)$  modeling the quantity of Cobalt-Thorium-G left in the sample after  $t$  days.

The known model for radioactive decay is an exponential decay function, so we know that our function should be  $f(t) = Ce^{kt}$  for constants  $C$  and  $k$  (we know  $k$  will be negative, because this is a decay function, but that information is not of immediate use to us). Furthermore, the described information about the material quantities at various times tells us that  $f(0) = 150$  (as the initial sample is 150g) and  $f(6) = 125$  (since six days later the sample has decayed to 125g). Thus we may easily determine  $C$ :

$$f(0) = 150 \quad \Rightarrow \quad Ce^0 = 150 \quad \Rightarrow \quad C = 150$$

and with a little more work, may determine  $k$ :

$$\begin{aligned} f(6) &= 125 \\ 150e^{k \cdot 6} &= 125 \\ e^{6k} &= \frac{125}{150} = \frac{5}{6} \\ 6k &= \ln \frac{5}{6} \\ k &= \frac{\ln \frac{5}{6}}{6} \end{aligned}$$

so our function should be  $f(t) = 150e^{\frac{\ln(5/6)}{6}t}$ .

- (b) **(4 points)** *How many days will it take the sample to decay to half of its original mass?*  
The sample will have halved in mass when  $f(t) = 75$ , so we want to find the value of  $t$  satisfying that equation:

$$\begin{aligned} 150e^{\frac{\ln(5/6)}{6}t} &= 75 \\ e^{\frac{\ln(5/6)}{6}t} &= \frac{75}{150} = \frac{1}{2} \\ \frac{\ln(5/6)}{6}t &= \ln \frac{1}{2} \\ \frac{\ln(5/6)}{6}t &= \frac{6 \ln \frac{1}{2}}{\ln(5/6)} \approx 22.81 \text{ days} \end{aligned}$$

The last approximation is given for illumination and could not easily be calculated under exam conditions.

- (c) **(3 points)** *How quickly, in grams per day, is the sample decaying 3 days into the experiment?*

The rate of decay at time  $t = 3$  is definitionally  $f'(3)$ . From the formula for  $f(t)$  above, we can easily compute  $f'(t)$ :

$$f'(t) = 150 \cdot \ln(5/6)6e^{\frac{\ln(5/6)}{6}t}$$

so  $f'(3) = 150 \cdot \ln(5/6)6e^{\frac{\ln(5/6)}{2}}$ . This is approximately  $-4.16$ , signifying that the sample is losing a mass of  $-4.16$  grams per day.

5. **(15 points)** *The cissoid of Diocles is a curve satisfying the equation  $x(x^2 + y^2) = 4y^2$ .*

- (a) **(12 points)** *Find a formula for  $\frac{dy}{dx}$  on this curve.*

Taking the derivative of each side, we get

$$\begin{aligned}\frac{d}{dx}(x(x^2 + y^2)) &= \frac{d}{dx}(4y^2) \\ (x^2 + y^2) + x\frac{d}{dx}(x^2 + y^2) &= \frac{dy}{dx}\frac{d}{dy}(4y^2) \\ (x^2 + y^2) + 2x^2 + x\frac{d}{dx}y^2 &= \frac{dy}{dx}8y \\ 3x^2 + y^2 + x\frac{dy}{dx}\frac{d}{dy}y^2 &= \frac{dy}{dx}8y \\ 3x^2 + y^2 + 2xy\frac{dy}{dx} &= 8y\frac{dy}{dx} \\ 2xy\frac{dy}{dx} - 8y\frac{dy}{dx} &= -3x^2 - y^2 \\ (2xy - 8y)\frac{dy}{dx} &= -3x^2 - y^2 \\ \frac{dy}{dx} &= \frac{-3x^2 - y^2}{2xy - 8y} = \frac{3x^2 + y^2}{8y - 2xy}\end{aligned}$$

(b) **(3 points)** Find the equation of the tangent line to the curve at  $(2, -2)$ .

At the specific values  $x = 2$ ,  $y = -2$ , we see from the above that

$$\frac{dy}{dx} = \frac{3 \cdot 2^2 + (-2)^2}{8(-2) - 2 \cdot 2(-2)} = \frac{16}{-8} = -2$$

Thus we have a line of the form  $y = -2x + b$ ; plugging in  $(2, -2)$  we can solve for  $b$ :

$$\begin{aligned}-2 &= -2 \cdot 2 + b \\ 2 &= b\end{aligned}$$

so  $y = -2x + 2$  is the tangent line.

6. **(8 points)** Estimate the following values using appropriate linear approximations.

(a) **(4 points)**  $1.03^5$ .

We consider the function  $f(x) = x^5$ , whose derivative is  $f'(x) = 5x^4$ . For  $x$  close to 1 (as 1.03 is), we can use the linear approximation:

$$f(x) \approx f(1) + (x - 1)f'(1)$$

Since  $f(1) = 1$  and  $f'(1) = 5$ , it follows that

$$f(1.03) \approx 1 + 0.03(5) = 1.15$$

For purposes of comparison, the actual value of  $1.03^5$  is 1.1592740743.

(b) **(4 points)**  $\sqrt{99.8}$ .

We consider the function  $f(x) = \sqrt{x}$ , whose derivative is  $\frac{1}{2\sqrt{x}}$ . For  $x$  close to 100 (as 99.8 is), we can use the linear approximation:

$$f(x) \approx f(100) + (x - 100)f'(100)$$

Since  $f(100) = 10$  and  $f'(100) = \frac{1}{2\sqrt{100}} = \frac{1}{20}$ , it follows that

$$f(99.8) \approx 10 - 0.2 \cdot \frac{1}{20} = 9.9$$

For purposes of comparison, the actual value of  $\sqrt{99.8}$  is around 9.989994995.

7. **(15 points)** *Amy is standing motionless 50 meters east of a north-south road with a radar gun, while Bob, who is 120 meters to the north, is driving south. The radar gun reports how quickly the distance between Amy and Bob is changing (which may not be Bob's actual speed).*

Our setup for this problem involves a right triangle between Bob's car, the place on the road parallel to Amy, and Amy's position well off the road. The east-west leg between Amy and the road is a constant length of 50, while the length of the north-south leg and the length of the hypotenuse are changing. We may denote these respectively by  $y$  and  $s$ , and note that, by the Pythagorean Theorem, they have the relationship  $y^2 + 50^2 = s^2$ . We also know that  $y$  is *currently* 120, and can compute in addition that  $s$  is currently

$$\sqrt{50^2 + y^2} = \sqrt{2500 + 14400} = \sqrt{16900} = 130$$

- (a) **(11 points)** *If Bob is driving south at 30 meters per second, what will the radar report as the rate of change of the distance between Bob and Amy?*

The information given to us here is that  $\frac{dy}{dt} = -30$  (since  $y$  is decreasing at a rate of 30 meters per second), and that we wish to find  $\frac{ds}{dt}$ . Using the relationship described in the general problem, we differentiate both sides with respect to  $t$ :

$$\begin{aligned} y^2 + 50^2 &= s^2 \\ \frac{d}{dt}(y^2 + 50^2) &= \frac{d}{dt}s^2 \\ \frac{dy}{dt} \frac{d}{dy}(y^2 + 50^2) &= \frac{ds}{dt} \frac{d}{ds}s^2 \\ \frac{dy}{dt} 2y &= \frac{ds}{dt} 2s \\ \frac{\frac{dy}{dt} y}{s} &= \frac{ds}{dt} \end{aligned}$$

and thus  $\frac{ds}{dt} = \frac{-30 \cdot 120}{130} = \frac{-36}{13}$ .

- (b) **(4 points)** *Conversely, if the radar reported a change-rate of 25 meters per second, what would Bob's actual speed be?*

In this case we are given  $\frac{ds}{dt} = \pm 25$ , and asked to find  $\frac{dy}{dt}$ . We may start with the second-to-last line of the derivation seen in the previous section:

$$\frac{dy}{dt} 2y = \frac{ds}{dt} 2s$$

and solve for  $\frac{dy}{dt} = \frac{\frac{ds}{dt} s}{y} = \frac{\pm 25 \cdot 130}{120} = \pm \frac{325}{12}$ .

8. **(10 points)** Find an equation of the tangent line to the curve  $y = e^x(x^2 - 4x + 3)$  at  $(0, 3)$ .

Using the product rule,

$$\frac{dy}{dx} = e^x(x^2 - 4x + 3) + e^x(2x - 4)$$

and specifically when  $x = 0$ ,  $\frac{dy}{dx} = 1(3) + 1(-4) = -1$ , so the desired line has slope  $-1$ . Since it passes through the point  $(0, 3)$ , it has point-slope form  $y - 3 = -1(x - 0)$ , or alternatively the slope-intercept form  $y = -x + 3$ .