1. **(16 points)** *Determine the domains of the following functions:*

   (a) **(4 points)** $f(x) = x^3 - 3x + \frac{1}{x}$.
   This function is evaluatable as long as the denominator of the fraction is nonzero; thus, the function's domain is the set of values where $x \neq 0$; in other words, simply when $x \neq 0$. In interval form, this would be $(-\infty, 0) \cup (0, \infty)$.

   (b) **(4 points)** $g(t) = \sqrt{9 - t^2}$.
   This function is evaluatable as long as the argument of the square root is non-negative; thus, the function's domain is the set of values where $9 - t^2 \geq 0$; in other words, when $t^2 \leq 9$, which is the case only when $|t| \leq 3$, or $-3 \leq t \leq 3$. In interval form, this would be $[-3, 3]$.

   (c) **(4 points)** $h(r) = \frac{\ln(r+4)}{r-3}$.
   This function is evaluatable as long as the argument of the logarithm is positive and the denominator of the fraction is nonzero; thus, the function's domain is the set of values where $r + 4 > 0$ and $r - 3 \neq 0$; in other words, when $r > -4$ and $r \neq 3$. In interval form, this would be $(-4, 3) \cup (3, \infty)$.

   (d) **(4 points)** $q(s) = \sqrt{8s + 3} - \ln(9 - s)$.
   This function is evaluatable as long as the argument of the square root is non-negative and the argument of the logarithm is positive; thus, the function's domain is the set of values where $8s + 3 \geq 0$ and $9 - s > 0$; in other words, when $s \geq \frac{-3}{8}$ and $s < 9$. In interval form, this would be $[\frac{-3}{8}, 9)$.

2. **(9 points)** *Let* $f(x) = 5x - 2$.

   (a) **(1 point)** *Find* $\lim_{x \to 3} f(x)$.
   Since $f(x)$ is a polynomial and thus continuous everywhere, direct substitution makes calculating this limit trivial: $\lim_{x \to 3} f(x) = f(3) = 5 \cdot 3 - 2 = 13$.

   (b) **(8 points)** *Using epsilon-delta methods, justify your result above.*
   Given a value of ϵ, we constrain $f(x)$ to be within ϵ of 13, and attempt to derive a sufficient bound on δ therefrom:

$$|5x - 2 - 13| < \epsilon$$
$$|5x - 15| < \epsilon \, |x - 3| < \frac{\epsilon}{5}$$

   So, since it is sufficient to require x within $\frac{\epsilon}{5}$ of 3, we may establish δ to be $\frac{\epsilon}{5}$.

3. **(10 points)** *Given the function* $f(x) = \frac{8-x^3}{x+2}$, *answer the following questions preparatory to sketching the functions.*

   (a) **(3 points)** *What is the domain of the function?*
   This function is evaluatable as long as the denominator of the fraction is nonzero; thus, the function's domain is the set of values where $x + 2 \neq 0$; in other words, when $x \neq -3$. In interval form, this would be $(-\infty, -2) \cup (-2, \infty)$.

(b) **(7 points)** *Describe, either in words or symbolically, the long-term behavior of the function in each direction.*

For very large positive, or large-magnitude negative, values of x, the comparative magnitudes of 8 and $-x^3$ will be such as to make 8 a negligible contribution to the numerator; likewise, in the denominator, $x+2$ will resemble x if x is of extraordinary magnitude. Thus, for very large-magnitude values of x, we see that $f(x) \approx \frac{-x^3}{x} = -x^2$, which is a quadratic function with negative coefficient. Thus, our reasonable understanding is that for x of very large magnitude, whether positive or negative, $f(x)$ will be of large magnitude and negative. Alternatively, one might write symbolically that

$$\lim_{x\to+\infty} f(x) = -\infty, \quad \lim_{x\to-\infty} f(x) = -\infty$$

4. **(13 points)** *Let $f(x) = 20 - 4x - x^2$.*

(a) **(9 points)** *Using the difference quotient, determine the formula for $f'(x)$.*

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{[20 - 4(x+h) - (x+h)^2] - (20 - 4x - x^2)}{h} \\
&= \lim_{h\to 0} \frac{(20 - 4x - 4h - x^2 - 2xh - h^2) - (20 - 4x - x^2)}{h} \\
&= \lim_{h\to 0} \frac{-4h - 2xh - h^2}{h} \\
&= \lim_{h\to 0} -4 - 2x - h \\
&= -4 - 2x
\end{aligned}$$

(b) **(4 points)** *Find the equation of the tangent line to $f(x)$ at the point $(3, -1)$.*

We know this line must pass through $(3, -1)$, with slope $f'(3) = -4 - 2\cdot 3 = -2$. Using point-slope form, we get

$$(y - (-1)) = -2(x - 3)$$

which can also be expressed in slope-intercept form as $y = -2x + 5$.

5. **(9 points)** *Let* $f(x) = \begin{cases} 3x & \text{if } x < 2 \\ x^2 + a & \text{if } 2 \le x < 8 \\ b\log_2 x & \text{if } x \ge 8 \end{cases}$.

What choices of a and b will make this function continuous?

Each of the individual parts of this function can be easily observed to be continuous on its domain, so problems can only arise at the junction points $x = 2$ and $x = 8$. To guarantee continuity at these points, we need to make sure that the left and right limits coincide, as such at $x = 2$:

$$\begin{aligned}
\lim_{x\to 2-} f(x) &= \lim_{x\to 2+} f(x) \\
\lim_{x\to 2-} 3x &= \lim_{x\to 2+} x^2 + a \\
6 &= 2^2 + a \\
2 &= a
\end{aligned}$$

And likewise for $x = 8$:

$$\begin{aligned}
\lim_{x\to 8-} f(x) &= \lim_{x\to 8+} f(x) \\
\lim_{x\to 8-} x^2 + a &= \lim_{x\to 8+} b\log_2 x \\
8^2 + 2 &= b\log_2 8 \\
66 &= 3b \\
22 &= b
\end{aligned}$$

So our solution is to choose $a = 2$ and $b = 22$.

6. **(20 points)** *Evaluate the following limits; when a limit can not be evaluated, explicitly say so.*

(a) **(4 points)** $\lim_{q\to-\infty} \frac{2q^3+q^2-4}{q^5-3q}$.

In the long term this function is dominated by its highest-degree terms in the numerator and denominator, so

$$\lim_{q\to-\infty} \frac{2q^3+q^2-4}{q^5-3q} = \lim_{q\to-\infty} \frac{2q^3}{q^5} = \lim_{q\to-\infty} \frac{2}{q^3} = 0$$

Alternatively, one may formally divide by the highest degree appearing in the denominator, although doing so is a bit more involved:

$$\lim_{q\to-\infty} \frac{2q^3+q^2-4}{q^5-3q} = \lim_{q\to-\infty} \frac{\frac{2q^3+q^2-4}{q^5}}{\frac{q^5-3q}{q^5}} = \lim_{q\to-\infty} \frac{\frac{2}{q^2}+\frac{1}{q^3}-\frac{4}{q^5}}{1-\frac{3}{q^4}} = \frac{0+0-0}{1-0} = 0$$

(b) **(4 points)** $\lim_{\theta\to 0} \frac{\sin\theta-2}{\cos\theta+3\sec\theta}$.

Since the numerator and denominator are continuous throughout their domains, we may use straightforward substitution as long as the denominator is nonzero, which indeed it is:

$$\lim_{\theta\to 0} \frac{\sin\theta-2}{\cos\theta+3\sec\theta} = \frac{\sin 0-2}{\cos 0+3\sec 0} = \frac{-2}{4} = \frac{-1}{2}.$$

(c) **(4 points)** $\lim_{t\to 4} \frac{t^2-8t+16}{t^2+t-20}$.

Direct substitution yields the incalculable form $\frac{0}{0}$, and we shall thus try to factor out $(t-4)$ from the numerator and denominator. Since we look near, but not at, $t = 4$, we can justify the straightforward cancellation arising therefrom:

$$\lim_{t\to 4} \frac{t^2-8t+16}{t^2+t-20} \lim_{t\to 4} \frac{(t-4)(t-4)}{(t-4)(t+5)} = \lim_{t\to 4} \frac{t-4}{t+5}.$$

Direct substitution into this new limit yields the result $\frac{0}{9} = 0$.

(d) **(4 points)** $\lim_{x\to\infty} \frac{8x^4-7x^2+2}{6-2x^4}$.

In the long term this function is dominated by its highest-degree terms in the numerator and denominator, so

$$\lim_{x\to\infty} \frac{8x^4-7x^2+2}{6-2x^4} = \lim_{x\to\infty} \frac{8x^4}{-2x^4} = -4.$$

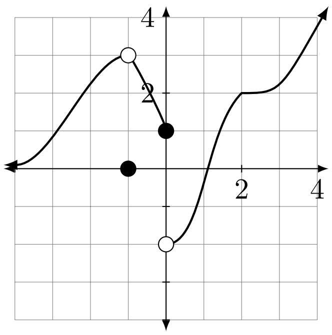
Alternatively, one may formally divide by the highest degree appearing in the denominator, although doing so is a bit more involved:

$$\lim_{x\to\infty} \frac{8x^4 - 7x^2 + 2}{6 - 2x^4} = \lim_{x\to\infty} \frac{\frac{8x^4-7x^2+2}{x^4}}{\frac{6-2x^4}{x^4}} = \lim_{x\to\infty} \frac{8 - \frac{7}{x^2} + \frac{2}{x^4}}{\frac{6}{x^4} - 2} = \frac{8 - 0 + 0}{0 - 2} = -4.$$

(e) **(4 points)** $\lim_{s\to 2} \frac{s^2+2s-1}{s^3-4s}$.

Direct substitution into this limit gives $\frac{7}{0}$, which, since it has a nonzero numerator and a zero denominator, demonstrates that the limit does not exist (in an asymptotic way).

7. **(7 points)** *For the plot of $h(x)$ shown below, indicate whether or not each of the following quantities can be evaluated. If they can be evaluated, compute their values. If they cannot be evaluated, explicitly say so. You need not show work.*



$\lim_{x\to -1^-} h(x) = 4$, since immediately to the left of the x-value -1, the graph is very close to the height $y = 4$.

$\lim_{x\to -1^+} h(x) = 4$, since immediately to the right of the x-value -1, the graph is very close to the height $y = 4$.

$h(-1) = 0$; note the solid dot at $(-1, 0)$.

$\lim_{x\to 0} h(x)$ does not exist, since in the vicinity of the x-value 2, the graph is takes on values close to two different heights: $y = 1$ on to the left, and $y = -2$ on the right. Since the behaviors on the two sides do not match, the two-sided limit cannot exist.

$h(0) = 1$; note the solid dot at $(0, 1)$.

$\lim_{x\to 2} h(x) = 2$, since in the vicinity of the x-value 2, the graph is very close to the height $y = 2$.

$h(2) = 2$; note that the graph passes through the point $(2, 2)$.

8. **(16 points)** *Calculate the following derivatives, using any method you wish.*

(a) **(4 points)** *Find* $\frac{d}{dt}\left((t^3 + 2t)(e^t - 4)\right)$.

The expression being differentiated is a product of two expressions, so it is best to use the product rule.

$$\begin{aligned}\frac{d}{dt}\left((t^3 + 2t)(e^t - 4)\right) &= \left(\frac{d}{dt}\left(t^3 + 2t\right)\right)\left(e^t - 4\right) + \left(t^3 + 2t\right)\frac{d}{dt}\left(e^t - 4\right) \\ &= \left(3t^t + 2\right)\left(e^t - 4\right) + \left(t^3 + 2t\right)e^t\end{aligned}$$

which need not be simplified further.

(b) **(4 points)** *If* $y = 3\sqrt[3]{x} - 8x^5 + \frac{4}{x^3} - 20$, *find* $\frac{dy}{dx}$.

If we rewrite $\frac{dy}{dx}$ as $\frac{d}{dx}(3x^{1/3} - 8x^5 + 4x^{-3} - 20)$ it is easy to use the power rule on each term in turn to get $x^{-2/3} - 40x^4 - 12x^{-4}$.

(c) **(4 points)** *Determine* $\frac{d}{dx}\frac{x^2+3e^x}{2x+1}$.

This is a quotient, so we use the quotient rule:

$$\begin{aligned} \frac{d}{dx}\frac{x^2+3e^x}{2x+1} &= \frac{(2x+1)\frac{d}{dx}(x^2+3e^x) - (x^2+3e^x)\frac{d}{dx}(2x+1)}{(2x+1)^2} \\ &= \frac{(2x+1)(2x+3e^x) - (x^2+3e^x)(2)}{(2x+1)^2} \end{aligned}$$

which need not be simplified further.

(d) **(4 points)** *For* $f(x) = 3x^2 + 4x - 8e^x + \frac{1}{x}$, *calculate its second derivative* $f''(x)$.

We may start by rewriting it so that each term is a power function or exponential function, and then use the appropriate differentiation rules twice:

$$\begin{aligned} f(x) &= 3x^2 + 4x - 8e^x + x^{-1} \\ f'(x) &= 6x + 4 - 8e^x - x^{-2} \\ f''(x) &= 6 - 8e^x + 2x^{-3} \end{aligned}$$