

1. **(12 points)** *A bar of iron which has been heated to 1400°F is taken from the furnace into a 100°F metalworking studio. After 5 minutes it has cooled to 800°F .*

- (a) **(5 points)** *Produce a function $T(t)$ modeling the bar's temperature t minutes after removal from the furnace.*

We know that this problem is modeled by Newton's Law of Cooling with an ambient temperature of 100°F , so our temperature model will be $T(t) = 100 + Ce^{-kt}$; it remains only to find C and k to have a final model.

Since the bar has a temperature of 1400°F immediately upon removal from the furnace, $T(0) = 1400$. Evaluating the left side of this equation, we find that $100 + Ce^0 = 1400$; thus $C = 1300$.

Since the bar has a temperature of 800°F five minutes later, we know that $T(5) = 800$. Expanding $T(5)$, we find that:

$$\begin{aligned} 100 + 1300e^{-k \cdot 5} &= 800 \\ 1300e^{-5k} &= 700 \\ e^{-5k} &= \frac{7}{13} \\ -5k &= \ln \frac{7}{13} \\ k &= \frac{-\ln \frac{7}{13}}{5} \end{aligned}$$

Assembling this value of k into our equation, we find that

$$T(t) = 100 + 1300e^{\frac{\ln \frac{7}{13}}{5}t}$$

- (b) **(3 points)** *How quickly is the bar's temperature changing immediately upon removal from the furnace?*

The time of the bar's removal from the furnace is definitionally time 0; thus the speed of the bar's cooling at that time is $T'(0)$. From the value of $T(t)$ above, we can easily compute $T'(t)$:

$$T'(t) = 1300 \frac{\ln \frac{7}{13}}{5} e^{\frac{\ln \frac{7}{13}}{5}t}$$

so $T'(0) = 1300 \frac{\ln \frac{7}{13}}{5} e^0 = 260 \ln \frac{7}{13}$. This is approximately -161 , signifying that the bar is cooling (dropping in temperature) by 161 degrees per minute.

- (c) **(4 points)** *The metal can be worked as long as it is hotter than 1000°F . How soon after the bar is removed from the furnace does it become unworkable?*

Since the metal becomes unworkable when $T(t) = 1000$, we want to find the value of t satisfying that equation:

$$\begin{aligned}
 100 + 1300e^{\frac{\ln \frac{7}{13}}{5}t} &= 1000 \\
 1300e^{\frac{\ln \frac{7}{13}}{5}t} &= 900 \\
 e^{\frac{\ln \frac{7}{13}}{5}t} &= \frac{9}{13} \\
 \frac{\ln \frac{7}{13}}{5}t &= \ln \frac{9}{13} \\
 t &= \frac{5 \ln \frac{9}{13}}{\ln \frac{7}{13}} \approx 3 \text{ minutes}
 \end{aligned}$$

2. **(10 points)** If $f(r) = \sin(2r^3 \ln r)$, calculate $f'(r)$.

Initial analysis suggests that we will be using both the chain rule (since $f(r)$ consists of the sine of a complicated expression), and the product rule (since that complicated expression is itself a product. We may define $u = 2r^4 \ln r$, and then:

$$\begin{aligned}
 f'(r) &= \frac{d}{dr} \sin(u) \\
 &= \frac{du}{dr} \frac{d}{du} \sin(u) \\
 &= \left(\frac{d}{dr} (2r^3 \ln r) \right) \cos(u) \\
 &= \left(\left(\frac{d}{dr} 2r^3 \right) \ln r + 2r^3 \frac{d}{dr} \ln r \right) \cos(u) \\
 &= \left(6r^2 \ln r + 2r^3 \cdot \frac{1}{r} \right) \cos(2r^4 \ln r) \\
 &= (6r^2 \ln r + 2r^2) \cos(2r^4 \ln r)
 \end{aligned}$$

3. **(8 points)** Estimate the following values using appropriate linear approximations.

- (a) **(4 points)** $\sqrt[3]{7.994}$.

We consider the function $f(x) = \sqrt[3]{x}$, whose derivative is $\frac{1}{3x^{2/3}}$. For x close to 8 (as 7.994 is), we can use the linear approximation:

$$f(x) \approx f(8) + (x - 8)f'(8)$$

Since $f(8) = 2$ and $f'(8) = \frac{1}{3 \cdot 8^{2/3}} = \frac{1}{12}$, it follows that

$$f(7.994) \approx 2 - 0.006 \cdot \frac{1}{12} = 1.9995$$

For purposes of comparison, the actual value of $\sqrt[3]{7.994}$ is around 1.999499875.

- (b) **(4 points)** $(-2.004)^5$.

We consider the function $f(x) = x^5$, whose derivative is $f'(x) = 5x^4$. For x close to -2 (as -2.004 is), we can use the linear approximation:

$$f(x) \approx f(-2) + (x + 2)f'(-2)$$

Since $f(-2) = -32$ and $f'(-2) = 80$, it follows that

$$f(-2.004) \approx -32 - 0.004(80) = -32.32$$

For purposes of comparison, the actual value of $(-2.004)^5$ is around -32.3212826 .

4. **(15 points)** *Hiro is motorcycling west at 60 mph from a point 120 miles east of the Black Sun, while Raven is 50 miles north of the Black Sun, driving northwards at 30 mph.*

(a) **(12 points)** *Are they growing closer together or further apart, and at what speed are they doing so?*

Let Hiro's distance to the west of the Black Sun in miles be x ; let Raven's distance to the north of the Black Sun in miles be y . Let s be the distance between them, and let t be time in hours. Since Hiro is driving *towards* the Sun, x is decreasing at 60 miles per hour; that is, $\frac{dx}{dt} = -60$; Raven is driving away, so y is increasing at 30 miles per hour, or $\frac{dy}{dt} = 30$. Furthermore, Hiro, the Black Sun, and Raven form a right triangle with hypotenuse of s , so by the Pythagorean Theorem

$$x^2 + y^2 = s^2$$

Since we want $\frac{ds}{dt}$, we differentiate this expression with respect to t to get:

$$\begin{aligned} 2x \frac{dx}{dt} + 2y \frac{dy}{dt} &= 2s \frac{ds}{dt} \\ \frac{2x \frac{dx}{dt} + 2y \frac{dy}{dt}}{2s} &= \frac{ds}{dt} \\ \frac{x \frac{dx}{dt} + y \frac{dy}{dt}}{s} &= \frac{ds}{dt} \end{aligned}$$

In the situation given here, $x = 120$ and $y = 50$ are given to us, and $s = \sqrt{120^2 + 50^2} = 130$, so $\frac{ds}{dt} = \frac{120(-60) + 50(30)}{130} = \frac{-570}{13}$. Thus the rivals are approaching each other (i.e. s is decreasing) at $\frac{570}{13}$ miles per hour.

(b) **(3 points)** *In an hour, will they be growing closer together or further apart, and at what speed will they be doing so?*

In the situation given here, $x = 120 - 60 = 60$ and $y = 50 + 30 = 80$ are given to us, and $s = \sqrt{60^2 + 80^2} = 100$, so $\frac{ds}{dt} = \frac{60(-60) + 80(30)}{100} = -12$. Thus the rivals are approaching each other (i.e. s is decreasing) at -12 miles per hour.

5. **(10 points)** *Find $\frac{d}{dt} \frac{e^{7t-3}}{\arctan(t^2)}$.*

Initial analysis suggests that we will be using both the quotient rule (since this expression is a quotient, on its outermost level), and the chain rule (since the arguments of the exponential and arctangent functions are themselves complicated expressions). We may pre-emptively

define $u = 7t - 3$ and $v = t^2$, and then:

$$\begin{aligned} \frac{d}{dt} \frac{e^u}{\arctan(v)} &= \frac{\arctan(v) \left(\frac{d}{dt} e^u \right) - e^u \frac{d}{dt} \arctan v}{\arctan(v)^2} \\ &= \frac{\arctan(v) \left(\frac{du}{dt} \frac{d}{du} e^u \right) - e^u \frac{dv}{dt} \frac{d}{dv} \arctan v}{\arctan(v)^2} \\ &= \frac{\arctan(v) (7e^u) - e^u \cdot 2t \cdot \frac{1}{1+v^2}}{\arctan(v)^2} \\ &= \frac{7 \arctan(t^2) e^{7t-3} - \frac{2te^{7t-3}}{1+(t^2)^2}}{\arctan(t^2)^2} \end{aligned}$$

6. (15 points) The conchoid of de Sluze is a curve satisfying the equation $(x-1)(x^2+y^2) = 4x^2$.

(a) (12 points) Find a formula for $\frac{dy}{dx}$ on this curve.

We differentiate both sides with respect to x , and use the product and chain rule as appropriate:

$$\begin{aligned} \frac{d}{dx} ((x-1)(x^2+y^2)) &= \frac{d}{dx} 4x^2 \\ \left(\frac{d}{dx} (x-1) \right) (x^2+y^2) + (x-1) \left(\frac{d}{dx} (x^2+y^2) \right) &= 8x \\ 1 \cdot (x^2+y^2) + (x-1)(2x+2yy') &= 8x \\ x^2+y^2 + (x-1)(2x) + (x-1)2yy' &= 8x \\ (x-1)2yy' &= 8x - x^2 - y^2 - (x-1)(2x) \\ y' &= \frac{10x - 3x^2 - y^2}{2(x-1)y} \end{aligned}$$

(b) (3 points) Find the equation of the tangent line to the curve at $(3,3)$.

At the specific values $x = 3$, $y = 3$, we see from the above that

$$\frac{dy}{dx} = \frac{10 \cdot 3 - 3 \cdot 3^2 - 3^2}{2(3-1)3} = \frac{-6}{12} = \frac{-1}{2}$$

Thus we have a line of the form $y = \frac{-x}{2} + b$; plugging in $(3,3)$ we can solve for b :

$$\begin{aligned} 3 &= \frac{-3}{2} + b \\ \frac{9}{2} &= b \end{aligned}$$

so $y = \frac{-x+9}{2}$ is the tangent line.

7. (20 points) Answer the following derivative-related questions.

- (a) **(6 points)** If $f(x) = (x^2 - 4x) \arcsin(x)$, find $f'(x)$.

This is an application of the product rule:

$$\begin{aligned} \frac{d}{dx} ((x^2 - 4x) \arcsin x) &= \frac{d}{dx} (x^2 - 4x) \arcsin x + (x^2 - 4x) \frac{d}{dx} \arcsin x \\ &= (2x - 4) \arcsin x + (x^2 - 4x) \cdot \frac{1}{\sqrt{1-x^2}} \end{aligned}$$

- (b) **(8 points)** Find $\frac{d}{dt} (\ln(\cos t))^5$.

This is an application of the chain rule in two stages, in which we shall let $u = \ln(\cos t) = \ln v$, and $v = \cos t$:

$$\begin{aligned} \frac{d}{dt} (\ln(\cos t))^5 &= \frac{d}{dt} u^5 \\ &= \frac{du}{dt} \frac{d}{du} u^5 \\ &= \left(\frac{d}{dt} \ln v \right) \frac{d}{du} u^5 \\ &= \frac{dv}{dt} \left(\frac{d}{dv} \ln v \right) \frac{d}{du} u^5 \\ &= \sin t \cdot \frac{1}{v} \cdot 5u^4 \\ &= \frac{5 \sin t (\ln(\cos t))^4}{\cos t} \end{aligned}$$

- (c) **(6 points)** If $y = \frac{e^x}{x^3 - \sqrt{x}}$, find $\frac{dy}{dx}$.

This is an application of the quotient rule:

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx} \frac{e^x}{x^3 - \sqrt{x}} \\ &= \frac{(x^3 - \sqrt{x}) \left(\frac{d}{dx} e^x \right) - e^x \frac{d}{dx} (x^3 - \sqrt{x})}{(x^3 - \sqrt{x})^2} \\ &= \frac{(x^3 - \sqrt{x})e^x - e^x(3x^2 - \frac{1}{2\sqrt{x}})}{(x^3 - \sqrt{x})^2} \end{aligned}$$

8. **(10 points)** Find an equation of the tangent line to the curve $y = \frac{x^2+3}{x^2+x+1}$ at $(2, 1)$.

Using the quotient rule,

$$\frac{dy}{dx} = \frac{(x^2 + x + 1)(2x) - (x^2 + 3)(2x + 1)}{(x^2 + x + 1)^2}$$

and specifically when $x = 2$, $\frac{dy}{dx} = \frac{7 \cdot 4 - 7 \cdot 5}{7^2} = \frac{-1}{7}$, so the desired line has slope $\frac{-1}{7}$, and thus equation $y = \frac{-x}{7} + b$. Since it passes through the point $(2, 1)$ we may specifically determine that $1 = \frac{-2}{7} + b$, and thus that $b = \frac{9}{7}$, for an equation $y = \frac{-x+9}{7}$ of the tangent line.