1. **(7 points)** *Let us consider the graph of the function $f(x) = x^4 - 2x^3$. Answer the following questions preparatory to sketching the curve.*

   (a) **(2 points)** *On what intervals of x-values is the function increasing? On which intervals is it decreasing? Label which is which.*

   We note that $f'(x) = 4x^3 - 6x^2 = (4x-6)x^2$. This is a polynomial, so it exists everywhere, but it is zero when either $4x - 6 = 0$ or $x^2 = 0$; in other words, it is zero at the values $x = \frac{3}{2}$ and $x = 0$. Among these points we can test different values to determine the sign of $f'(x)$:

   $$f'(-1) = 4(-1)^3 - 6(-1)^2 = -10, \text{ so } f'(x) \text{ is negative on } (-\infty, 0)$$
   $$f'(1) = 4 \cdot 1^3 - 6 \cdot 1^2 = -2, \text{ so } f'(x) \text{ is negative on } (0, \frac{3}{2})$$
   $$f'(2) = 4 \cdot 2^3 - 6 \cdot 2^2 = 8, \text{ so } f'(x) \text{ is positive on } (\frac{3}{2}, \infty)$$

   Thus, $f(x)$ is decreasing when $x < \frac{3}{2}$ (except momentarily at $x = 0$) and increasing when $x > \frac{3}{2}$.

   (b) **(2 points)** *At what x-values are the local extrema, and which type of local extremum is each?*

   As noted above, $f'(x)$ has two zeroes which are critical points, $x = 0$ and $x = \frac{3}{2}$. Since $x = 0$ is not a transition in sign of $f'(x)$ (it touches zero between two bouts of negative values) it is not actually an extremum. On the other hand, $x = \frac{3}{2}$ is a transition from decrease to increase, and is thus a local minimum.

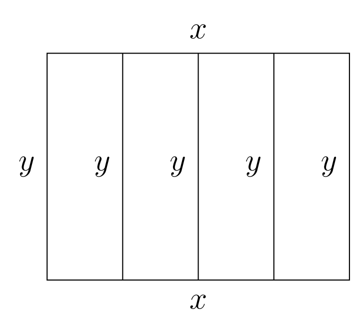
   (c) **(3 points)** *On what intervals of x-values is the function concave up? On which is it concave down? Label which is which. Also identify the points of inflection.*

   We note that $f''(x) = 12x^2 - 12x = 12(x-1)x$. This is a polynomial, so it exists everywhere, but it is zero when either $x = 0$ or $x = 1$. Among these points we can test different values to determine the sign of $f''(x)$:

   $$f'(-1) = 12(-1)^2) - 12(-1) = 24, \text{ so } f''(x) \text{ is positive on } (-\infty, 0)$$
   $$f'(\tfrac{1}{2}) = 12(\tfrac{1}{2})^2) - 12(\tfrac{1}{2}) = -3, \text{ so } f''(x) \text{ is negative on } (0, 1)$$
   $$f'(2) = 12(2)^2) - 12(2) = 24, \text{ so } f''(x) \text{ is positive on } (1, \infty)$$

   Thus, $f(x)$ is concave up when $x < 0$ or $x > 1$, and concave down when $0 < x < 1$, and points of inflection appear at the transitional points $x = 0$ and $x = 1$.

2. **(7 points)** *We want to enclose a rectangular animal pasture with a fence all around the outside as well as three fences parallel to a pair of sides of the pasture, so as to divide the enclosure into four rectangular sections. We have 1000 feet of fencing to use. What dimensions for our pasture maximize its area?*



The above drawing is a representation of the scenario described; we assign the two dimensions of the field the labels of x and y; note that each of the seven fences then have lengths of x or y.

Our goal is to maximize the area of the field, subject to the condition that the total quantity of fencing used is 1000 feet. Since the total length of fencing can be seen to be $2x+5y$, and the area of the field is xy, these correspond to the problem of maximizing xy subject to the constraint that $2x + 5y = 1000$. We may re-express this constraint as $y = \frac{1000-2x}{5} = 200 - \frac{2}{5}x$, so that the expression we seek to maximize is, in a single variable, $A(x) = x(200 - \frac{2}{5}x) = 200x - \frac{2}{5}x^2$. The interval of acceptable values on x is $(0, 500]$, since we can use no more than 1000 feet even on the two fences of length x.

Since $A'(x) = 200-\frac{4}{5}x$, we see that $A(x)$ will have one critical point at $x = 250$. Our candidates for optimization are thus the interval endpoints $x = 0$ and $x = 450$ together with this critical point 250. Since $A(0) = 0$ and $A(500) = 0$, the critical point 250 (at whech the area is actually positive) must be optimal. Thus, our area-maximizing choice of x is 250, which has an associated value of $y = 200 - \frac{2}{5} \cdot 250 = 100$, so our optimal dimensions are 250×100.

3. **(6 points)** *Determine the value of the following limits.*

(a) $\lim_{x\to+\infty} \frac{(\ln x)^2}{x}$.

As x grows very large, so do both $(\ln x)^2$ and x, so this ratio will be a $\frac{\infty}{\infty}$ indeterminate form, to which we can apply L'Hôpital's rule. In the numerator we must use either the product rule or the chain rule to determine that $\frac{d}{dx}(\ln x)^2 = \frac{2}{x}\ln x$, and we find that

$$\lim_{x\to+\infty} \frac{(\ln x)^2}{x} = \lim_{x\to+\infty} \frac{\frac{2}{x}\ln x}{1} = \lim_{x\to+\infty} \frac{2\ln x}{x}$$

However, since $\ln x$ and x are both growing very large as x grows large, we still have a $\frac{\infty}{\infty}$ indeterminate form, to which we should apply L'Hôpital's rule again.

$$\lim_{x\to+\infty} \frac{2\ln x}{x} = \lim_{x\to+\infty} \frac{\frac{2}{x}}{1} = \lim_{x\to+\infty} \frac{2}{x} = 0$$

(b) $\lim_{\theta\to 0} \frac{\sin\theta-\theta}{\theta^3}$.

Evaluation at $\theta = 0$ yiels zero for both the numerator and denominator, so this is a $\frac{0}{0}$ indeterminate form, to which L'Hôpital's rule can be applied:

$$\lim_{\theta\to 0} \frac{\sin\theta - \theta}{\theta^3} = \lim_{\theta\to 0} \frac{\cos\theta - 1}{3\theta^2}$$

but this still evaluates to $\frac{0}{0}$, requiring another invocation of L'Hôpital's rule:

$$\lim_{\theta\to 0} \frac{\cos\theta - 1}{3\theta^2} = \lim_{\theta\to 0} \frac{-\sin\theta}{6\theta}$$

which is, alas, still evaluating to $\frac{0}{0}$, so we use L'Hôpital's rule once more:

$$\lim_{\theta\to 0} \frac{-\sin\theta}{6\theta} = \lim_{\theta\to 0} \frac{-\cos\theta}{6} = \frac{-1}{6}$$

(c) $\lim_{t\to 0} \frac{e^t}{t\sin t+\cos t}$.

Direct evaluation here yields $\frac{e^0}{0\sin 0+\cos 0} = \frac{1}{1} = 1$, so no interesting tricks are necessary.