1. **(5 points)** *Determine how many 4-letter strings there are using the letters "A", "B", and "C" such that each letter appears at least once in the string.*

   We consider all 81 possible strings in lexicographic order, only writing down those meeting the given conditions.

AABC	ABCC	BAAC	BBCA	CAAB	CBAB
AACB	ACAB	BABC	BCAA	CABA	CBAC
ABAC	ACBA	BACA	BCAB	CABB	CBBA
ABBC	ACBB	BACB	BCAC	CABC	CBCA
ABCA	ACBC	BACC	BCBA	CACB	CCAB
ABCB	ACCB	BBAC	BCCA	CBAA	CCBA

   for a total of 36 strings.

2. **(5 points)** *Determine how many ways there are to make change for \$1.00 using only quarters, nickels, and pennies. You may find it simpler if you use ellipsis to avoid having to write out the middle of long patterns.*

   We shall examine ways in order from using the most of the largest-denomination coin to the least: thus we might start by considering the distributions using four quarters, then three, then two, then one, then zero; within each of these cases we will distribute the two remaining denominations in the same manner, considering, e.g., within the two-quarter case everything from ten nickels down to zero nickels. Using such an ordering, we find the following list (note that in some places an obvious pattern has been elided, with counts noted):

Quarters	Nickels	Pennies
4	0	0
3	5	0
3	4	5
3	3	10
3	2	15
3	1	20
3	0	25
2	10	0
2	9	5
	⋮ (11 rows)	
2	0	50

Quarters	Nickels	Pennies
1	15	0
1	14	5
	⋮ (16 rows)	
1	0	75
0	20	0
0	19	5
	⋮ (21 rows)	
0	0	100

   for a total of $1 + 6 + 11 + 16 + 21 = 55$ ways of making change.

3. **(5 points)** *You are making up little gift bags containing 5 candies of 3 different types: chocolates, caramels and toffees, and you can use as many as you like of each type*

of candy. Inside the bag, of course, the candies have no intrinsic order. How many different types of gift bags are possible (e.g., "2 chocolates, 1 caramel, and 2 toffees", is one type, and "no chocolates, 4 caramels, and 1 toffee" would be another)?

We might arbitrarily come up with an ordering for the different possible bags based first on the number of chocolates, then on the number of caramels, and then the number of toffees. So we'd start with the gift bags with 5 chocolates, and work our way down to those with no chocolates.

Chocolates	Caramels	Toffees
5	0	0
4	1	0
4	0	1
3	2	0
3	1	1
3	0	2
2	3	0
2	2	1
2	1	2
2	0	3

Chocolates	Caramels	Toffees
1	4	0
1	3	1
1	2	2
1	1	3
1	0	4
0	5	0
0	4	1
0	3	2
0	2	3
0	1	4
0	0	5

4. **(5 points)** *How many ways are there to express* 8 *as an unordered sum of odd numbers? (e.g., "7+1" is one way, and is the same as "1+7"; "1+1+1+1+1+1+1+1" is a different way).*

   Since the sum is unordered, we might, for purposes of making certain we count each one only once, write our sums starting with the largest term and working our way down to the smallest.

   The largest value possible in sums is 7, so we might start with those sums containing a "7". There's only one: $7 + 1$.

   Next we might consider those sums containing at least one "5" and no "7": Such a sum has a "5" as well as other odd numbers adding up to 3, and there are two ways to do that: $5 + 3$ or $5 + 1 + 1 + 1$.

   Next let us consider sums containing at least one "3" and no "5" or "7". We could have $3 + 3 + 1 + 1$, or $3 + 1 + 1 + 1 + 1 + 1$.

   Finally, there is exactly one sum using only terms of the form "1": $1 + 1 + 1 + 1 + 1 + 1 + 1 + 1$.

   There are thus six ways to write this sum.

5. **(10 points)** *Determine how many three-digit numbers (integers between 100 and 999, inclusive) there are in which all three digits are of the same parity (either all odd or all even). For instance, 402 is such a number, but 363 is not.*

   Let us separately count those numbers in which all digits are odd, and those in which all digits are even. If all the digits are odd, then there are five choices for each digit, totaling $5 \times 5 \times 5 = 125$ ways to build a three-digit number with all odd digits, while if all the digits are even, there are only four choices for the first digit (since zero is not allowed) and five for each of the remaining two, yielding $4 \times 5 \times 5 = 100$ ways to build a three-digit nubmer with all even digits. Thus, in total there are $125 + 100 = 225$ three-digit numbers in which all the digits have the same parity.

6. **(10 points)** *A modified deck of cards contains five suits and ten cards in each suit . A "flush" is defined as in poker as a hand in which all cards are from the same suit (ignore, for the purposes of this problem, the poker convention that a "straight flush" or "royal flush" would be a different type of hand). How many different five-card hands (which are not ordered) are flushes? What is the probability that a random hand is a flush?*

   A "flush" hand can be built by selecting a suit, and then selecting five cards within that suit. There are five ways to select a suit, and $\binom{10}{5}$ ways to select cards within that suit, so there are a total of $5 \times \binom{10}{5} = \frac{5\times10\times9\times8\times7\times6}{5\times4\times3\times2\times1} = 1260$ flushes. Since there are a total of $\binom{50}{5} = \frac{50\times49\times48\times47\times46}{5\times4\times3\times2\times1} = 2118760$ hands, the probability of a random hand being a flush is $\frac{1260}{2118760} = \frac{9}{15134} \approx 0.06\%$.

7. **(10 points)** *Find a formula for the number of ways to build an ordered string of four letters drawn from an alphabet consisting of n letters, such that the string does not contain the same letter exactly three times.*

   In order to contain the same letter exactly three times, a string of four letters must conform to one of the four templates AAAB, AABA, ABAA, or BAAA for different letters A and B. There are n possible letters to stand in for A and $n-1$ to stand in for B; thus, there are $4n(n-1)$ strings in which the same letter *does* occur exactly three times. There are n^4 strings in total, so there are $n^4 - 4n(n-1)$ in which the same letter does not occur three times.

Musica est exercitium arithmeticae occultum nescientis se numerare animi.
—Gottfried Leibniz