Finding and using appropriate statistics

You can use either a standard combinatorial statistic (exponents, binomials, multinomials, etc.), or a combination of multiple combinatorial statistics to solve these problems; explain why you make the choices you do in solving the problem.

1. **(6 points)** *How many solutions in positive integers are there to the equation* $x+y+z = 20$*, if* z *must be less than 10?*

   This could be construed as the number of ways to distribute 20 blank balls to three labeled boxes (representing the values of x, y, and z) such that each box receives one ball, and furthermore the "Z" box receives no more than 9.

   We can take a difference of two statistics to do this: we start with the distributions of 20 balls to three boxes with at least one ball per box. We pre-emptively assign 3 balls, and the remaining 17 are placed by a balls-and-walls paradigm in any of $\binom{19}{2}$ ways.

   From this count we then must *subtract* those distributions which assign 10 or more balls to box Z. To count these, we pre-emptively assign one ball to each of boxes X and Y, as before, and assign 10 balls to box Z, leaving 8 balls to be distributed by a balls-and-walls paradigm in any of $\binom{10}{2}$ ways.

   Thus, there are a total of $\binom{19}{2} - \binom{10}{2} = 126$ ways to distribute the balls, and 126 associated solutions.

2. **(6 points)** *I have 4 unique coins (a Double Eagle, a Seated Liberty Dollar, a Half Dime, and a Twenty Cent Piece), as well as 10 identical coins (Indian Head cents). I want to give these away to my 3 numismatic friends. How many ways are there to do so, if I may choose to give each friend as many or as few coins as I wish?*

   There might be multiple ways to do this, but one straightforward approach would be to break this down into the separate calculations associated with the number of ways to distribute the unique and non-unique objects.

   We first calculate the number of ways to distribute the unique objects: each of the four can be chosen to go to any of 3 recipients, so we make a choice among 3 alternatives four times, and thus have $3^4 = 81$ ways to distribute these items.

   Then we distribute the 10 non-unique items among our 3 (presumably distinguishable) friends, and according to a balls-and-walls paradigm we can consider this to be the same as choosing places for two partitions among 12 slots; this can be done in $\binom{12}{2} = 66$ ways.

   Thus, in total, we can perform these a distribution procedure composed of these two subprocedures in any of $81 \times 66 = 5346$ ways.

3. **(8 points)** *Find something counted by the formula* $\binom{n}{k}\binom{k}{\ell}$*. Explain why you could count the same thing in a different way by using the formula* $\binom{n}{\ell}\binom{n-\ell}{k-\ell}$*.*

   We could naïvely interpret $\binom{n}{k}$ as counting the k-element subsets of $\{1, 2, \ldots, k\}$, and $\binom{k}{\ell}$ as the number of ℓ-element subsets of that particular cohosen subset. Thus, the structure this product counts is the selection of a k-element subset S of $\{1, 2, \ldots, n\}$ and a subset T of S; this is, after a fashion, a multi-element version of our "distinguished-element" count method in class.

But we could also count these objects by building T first, and then constructing the additional elements of S around it! Since T consists of ℓ elements of $\{1, 2, \dots, n\}$, the set T could be constructed in any of $\binom{n}{\ell}$ ways. Now, to build S, we need to take T together with $k-\ell$ additional elements. There is a pool of $n-\ell$ elements not already in S to choose from, so we could build up these additional elements of S in any of $\binom{n-\ell}{k-\ell}$ ways, so the number of ways to build the aforementioned structure is also equal to $\binom{n}{\ell}\binom{n-\ell}{k-\ell}$.

4. **(10 points)** *Find something counted by the formula* $\sum_{i=0}^{k}\binom{n}{i}\binom{m}{k-i}$. *Explain why you could count the same thing in a different way by using the formula* $\binom{n+m}{k}$.

   One obvious way to think about $\binom{n+m}{k}$ is as a count of the k-element subsets of $\{1, 2, \dots, n+m\}$, and the summation suggests that we might instead want to think of n-element and m-element sets, so let's consider what might happen to a k-element subset of $\{1, 2, \dots, n+m\}$ if we split the host set into $\{1, 2, \dots, n\}$ and $\{n+1, n+2, \dots, n+m\}$. Some of the elements might appear in the first part, while others would appear in the second part. If we wanted to count the number of ways to build a k-element set by selecting some elements from each of $\{1, 2, \dots, n\}$ and $\{n+1, n+2, \dots, n+m\}$, we might consider how many such sets could be build with i elements in the first part. Then we select i elements of $\{1, 2, \dots, n\}$ in any of $\binom{n}{i}$ ways, and select $k-i$ elements of $\{n+1, n+2, \dots, n+m\}$ in any of $\binom{m}{k-i}$ ways, for a total of $\binom{n}{i}\binom{m}{k-i}$ ways of selecting k elements with i in the first part. Since plausibly anywhere from 0 to k elements could occur in the first part, to get the total number of k-element subsets of $\{1, 2, \dots, n+m\}$, we need to add up the expression $\binom{n}{i}\binom{m}{k-i}$ for each of the cases from $i = 0$ to $i = k$, giving the hoped-for count $\sum_{i=0}^{k}\binom{n}{i}\binom{m}{k-i}$.

5. **(10 points)** *Explain why it must be the case that if we choose 271 positive integers, four of them must have the same first digit and the same last digit.*

   Let us build 90 "boxes" for positive integers, classifying them by their first digit (any of 9 possibilities) and their last digit (any of 10). So we would have, for instance, a "3—5" box which would be the natural home for 35, 305, 315, 325, and so forth (including, for example, 372015), or a "7—7" box, which is the natural home for 7, 77, 707, 717, and so forth. Every positive integer has a home, and there are 90 homes in total. Thus, when we classify 271 positive integers, the pigeonhole principle guarantees that four of them end up in the same box (since $90 \times 3 = 270 < 271$), and by the nature of our boxes, these four numbers have the same first and last digits.

6. **(10 points)** *How many ways are there to arrange the numbers from 1 to 6 so that* at least one *number appears immediately before the next largest number? (e.g., 156342 would be such an arrangement, since it has a "56" and a "34").*

   Let A_i be the set of permutations in which i appears immediately before $i+1$. then all we want is the union of the sets A_1, A_2, A_3, $\dots$, A_5. The inclusion-exclusion principle allows us to count this by adding their sizes, then subtracting the sizes of pairwise intersections, and adding the size of their triplewise intersections, and so forth.

We might note that in order to find $|A_1|$, we could consider permuting 5 elements, one of which is the "glued-together" string "12", and the rest of which are the numbers 3, 4, 5, and 6. There are 5! ways to order this. A similar argument with different 'glued-together" pieces shows that the other A_i also have size 5!.

If we looked at the sizes of intersections $A_i \cap A_j$, we might note that there are two different cases: if $j = i+1$, then we glue together a triple "$i(i+1)(i+2)$", while otherwise we have two glued-together blocks of two: "$i(i+1)$" and "$j(j+1)$". However, in either case we have 4 objects in total being rearranged, in 4! possible ways.

Likewise, we might note that each triple intersection describes rearrangements of the numbers with three "gluings-together" reducing the number of objects to three and giving 3! possible rearrangements; quadruple and quintuple intersections have, in a similar fashion, 2! and 1! elements respectively. We now apply the principle of inclusion-exclusion to get:

$$5 \cdot 5! - \binom{5}{2} \cdot 4! + \binom{5}{3} \cdot 3! - \binom{5}{4} \cdot 2! + \binom{5}{5} \cdot 1! = 411$$

> And NUH is the letter I use to spell Nutches
> Who live in small caves, known as Nitches, for hutches
> These Nutches have troubles, the biggest of which is
> The fact there are many more Nutches than Nitches.
>
> —Dr Seuss, *On Beyond Zebra*