Ordinary Generating Functions

The following questions pertain to the construction and use of ordinary generating functions.

1. **(10 points)** For each of the following sequences, give the ordinary generating function of the sequence in a form without any summation notation; i.e., simply as a rational function or something easily converted to a rational function.

   (a) **(5 points)** Let a_n represent the number of ways to distribute n identical coins to Alice, Bob, and Carla such that Alice gets no more than 3, Bob gets no fewer than 2, and Carla receives between 5 and 10.

   (b) **(5 points)** Let b_n represent the number of ways to distribute n balls to 4 boxes so that either every box contains an even number of balls or every box contains an odd number of balls.

2. **(10 points)** By isolating coefficients of your generating function, find a formula for a_n in question 1(a).

Partitions

The following questions pertain to enumerating partitions; note that both generating functions and bijections between families of generating functions are used here.

3. **(10 points)** Explain, either with generating function equality or with an appropriate bijection, why it is true that the number of partitions of n into k distinct parts is equal to the number of partitions of n into the parts $(1, 2, 3, \ldots, k)$, with each part being selected at least once. For instance, the partitions of 9 into 3 distinct parts are $6+2+1$, $5+3+1$, and $4+3+2$, while the partitions of 9 into parts from $(1, 2, 3)$ are $3+3+2+1$, $3+2+2+1+1$, and $3+2+1+1+1+1$; these can both be done in 3 ways.

4. **(5 point bonus)** Using generating functions, prove that the partition of n into odd parts is equal to the number of partitions of n into distinct parts.

5. **(10 points)** Describe a bijection between those partitions of n which contain at least one 1 and those in which the largest part is unique. For isntance, 6 has sevel partitions in which the largest part is unique: 6, $5+1$, $4+2$, $4+1+1$, $3+2+1$, $3+1+1+1$, $2+1+1+1+1$ and seven which include a 1: $5+1$, $4+1+1$, $3+2+1$, $3+1+1+1$, $2+2+1+1$, $2+1+1+1+1$, $1+1+1+1+1+1$.

Exponential Generating Functions

The following questions pertain to the construction of exponential generating functions.

6. **(10 points)** Find an exponential generating function for each of the following sequences:

   (a) **(5 points)** a_n, where a_n is the number of n-letter strings in which each vowel is used *no more than* once.

   (b) **(5 points)** b_n, where b_n is the number of n-letter strings in which each vowel is used *at most* once.