1. **(6 points)** *Find (but do not solve) a homogeneous linear recurrence relation, including initial conditions, for the number a_n of ways to tile a $2 \times n$ rectangle with any number of red, blue, and green dominoes.*

   Clearly, $a_0 = 1$ (since the empty rectangle has a unique empty tiling), and $a_1 = 3$, since a 2×1 rectangle can be tiled with a single domino of any of 3 colors. Thenceforth, we can tile a $2 \times n$ rectangle for $n \geq 2$ with one of two options, described below.

   First, we could have a vertical domino in any of three colors at the left side of the rectangle, leaving a $2 \times (n-1)$ rectangle left to tile in any of a_{n-1} ways; this sort of tiling can thus be achieved in any of $3a_{n-1}$ ways.

   Alternatively, we could have a horizontal domino in the lower left; then the 1×2 space above it *must* be filled with another domino. These two dominoes could have any of $3^2 = 9$ color combinations, and they leave a $2 \times (n-2)$ rectangle left to be filled, so there are $9a_{n-2}$ ways to tile a rectangle this way.

   Taking these two types of tiling together, we see that $a_n = 3a_{n-1} + 9a_{n-2}$.

2. **(7 points)** *Find (but do not solve) a nonhomogeneous linear recurrence relation, including initial conditions, for the number b_n of ternary strings (strings of the numbers 0, 1, and 2) which* contain *the subsequence "01".*

   Our initial conditions will in fact be $b_0 = b_1 = 0$, since the requested inclusion is impossible for a string of length less than 2. In building a recurrence, there are two possibilities to consider: a string of length n may begin with 01, in which case the remaining $n-2$ digits could be anything, so there are 3^{n-2} such strings.

   We can now count the strings which have a "01" somewhere after the first digit. The first digit could be any of three things, and the remainder can be constructed in any of b_{n-1} ways, for a possibility of $3b_{n-1}$ such strings.

   However, now we have double-counted those which both begin with "01" and have a later "01"! There will be b_{n-2} such strings, since they start with a "01", and have another "01" in the string of length $n-2$ following.

   Thus, we get that $b_n = 3^{n-2} + 3b_{n-1} - b_{n-2}$.

3. **(7 points)** *For the sequence a_n from question 1, find a rational-function representation of the ordinary generating function $\sum_{n=0}^{\infty} a_n z^n$.*

   Adding up multiples of the recurrence to build something akin to the generating function, we have:

$$\sum_{n=2}^{\infty} a_n z^n = \sum_{n=2}^{\infty} (3a_{n-1} + 9a_{n-2}) z^n$$

which we can manipulate to isolate the generating function:

$$\begin{aligned}
\sum_{n=2}^{\infty} a_n z^n &= 3z \sum_{n=2}^{\infty} a_{n-1} z^{n-1} + 9z^2 \sum_{n=2}^{\infty} a_{n-2} z^{n-2} \\
\sum_{n=0}^{\infty} a_n z^n - a_1 z - a_0 &= 3z \left(\sum_{n=0}^{\infty} a_n z^n - a_0 \right) + 9z^2 \sum_{n=0}^{\infty} a_n z^n \\
\sum_{n=0}^{\infty} a_n z^n - 3z - 1 &= 3z \sum_{n=0}^{\infty} a_n z^n - 3z + 9z^2 \sum_{n=0}^{\infty} a_n z^n \\
(1 - 3z - 9z^2) \sum_{n=0}^{\infty} a_n z^n &= 1 \\
\sum_{n=0}^{\infty} a_n z^n &= \frac{1}{1 - 3z - 9z^2}
\end{aligned}$$

4. **(10 points)** *Find a solution to the recurrence relation $a_n = 7a_{n-2} + 6a_{n-3}$ with the initial conditions $a_0 = 9$, $a_1 = 10$, and $a_2 = 32$, using whichever method you wish.*

   The characteristic equation for this recurrence is $\lambda^3 - 7\lambda + 6 = 0$, which has roots 1, -2, and -3, so the general solution to the recurrence is $a_n = k + \ell \cdot 2^n + m(-3)^n$. Plugging in the initial conditions, we can solve for k, ℓ, and m:

$$\begin{cases} 9 = k + \ell + m \\ 10 = k + 2\ell - 3m \\ 32 = k + 4\ell + 9m \end{cases}$$

   and taking a pairwise difference, we can eliminate k:

$$\begin{cases} 1 = \ell - 4m \\ 22 = 2\ell + 12m \end{cases}$$

   and subtracting twice the first equation from the second gives $20m = 20$, so $m = 1$, and then $\ell = 1 + 4m = 5$, and $k = 9 - \ell - m = 3$, for the final closed form $a_n = 3 + 5 \cdot 2^n + (-3)^n$.

5. **(10 points)** *Find a solution to the recurrence relation $a_n = 2a_{n-1} - a_{n-2}$ with the initial conditions $a_0 = 4$ and $a_1 = 1$.*

   The characteristic equation for this recurrence is $\lambda^2 - 2\lambda + 1 = 0$, which has 1 as a root of multiplicity 2, so the general solution is $a_n = k + \ell n$. Simple solution of this linear equation shows that $a_n = 4 - 3n$.

6. **(10 points)** *Recall from class the idea of a "growing annuity" where in the ith year you withdraw i dollars: we saw there that this concept with, say, an interest rate of 4% is modeled by the recurrence $a_n = 1.04a_{n-1} - n$. Solve this recurrence in terms of the initial investment a_0. How large would a_0 have to be in order for this annuity to be permanently self-perpetuating, i.e. never run out of money?*

We will need to find the general solution to the linear inhomogeneous recurrence $a_n = 1.04a_{n-1} - n$, and then explore its long-term behavior. The associated homogeneous equation $b_n = 1.04b_{n-1}$ has the straightforward solution $b_n = k(1.04)^n$.

Now, in seeking a particular solution to the original inhomogeneous equation, we note that the inhomogeneous component is a linear function in n, so our template for the partipular solution will be $a_n^p = An + B$. Plugging this into our recurrence:

$$\begin{aligned} a_n^p &= 1.04a_{n-1}^p - n \\ An + B &= 1.04(A(n-1) + B) - n \\ An + B &= (1.04A - 1)n + (-1.04A + 1.04B) \end{aligned}$$

so in order for this equality to be universally true, the linear and constant terms must be equal:

$$\begin{cases} A = 1.04A - 1 \\ B = -1.04A + 1.04B \end{cases}$$

The first equation, when solved, gives $A = 25$, and plugging this into the second gives $0.04B = 1.04A = 26$, so $B = 650$.

Our general solution is thus that $a_n = b_n + a_n^p = k(1.04)^n + 25n + 650$. Plugging in $a_0 = k + 650$ gives $k = a_0 - 650$ for a solution $a_n = (a_0 - 650)(1.04)^n + 25n + 650$.

Over the long term, this function's behavior is dominated by the exponential term: if its coefficient is positive, it will increase rapidly without bound, if negative, it will decrease without bound (and, particulalrly, must be negative), and if zero, it will exhibit modest, slow growth. Thus, we see our breakeven is when $a_0 = 650$, so an initial investment of \$650 will yield a perpetual annuity.

7. **(5 point bonus)** *For F_n the Fibonacci numbers (with $F_0 = 1$ and $F_1 = 1$), show that F_n and $\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i}$.*

   F_n counts the number of ways to tile a $1 \times n$ rectangle with dominoes and checkers. We might divide this into cases based on the number i of dominoes (which can be no more than $\lfloor \frac{n}{2} \rfloor$). If we have i dominoes, then we would require $n - 2i$ checkers to tile the rectangle. We can consider that we thus now have $n - i$ objects, i of which are dominoes, to order. We can thus select i positions, out of the $n - i$ possible positions in the ordered arrangement of objects, to contain dominoes, in $\binom{n-i}{i}$ ways. Thus, there are $\binom{n-i}{i}$ ways to tile a $1 \times n$ rectangle with i dominoes and $n - 2i$ checkers. Adding up over all choices of i gives us a total of $\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i}$ ways to tile a $1 \times n$ rectangle with any number of dominoes and checkers, which, as was seen above, is equal to F_n.

So, naturalists observe, a flea
Has smaller fleas that on him prey,
And these have smaller still to bite 'em,
And so proceed *ad infinitum.*
—Jonathan Swift, "On Poetry: A Rhapsody"