Algorithm Design and Analysis

1. **(4 points)** Clearly x^2 can be calculated with a single multiplication; explain how x^4 can be calculated with only two multiplications, x^8 with only three, x^{16} with four, and so forth.

2. **(10 points)** Using the above technique, construct a logarithmic-time algorithm (that is, an algorithm whose number of operations is approximately proportional to $\log n$) for calculating x^n.

3. **(10 points)** Build an algorithm to take the dot product of two vectors $\vec{u}$ and $\vec{v}$, which takes as input the values $u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n$, and outputs their dot product. Determine the asymptotic runtime of this algorithm.

4. **(5 points)** A matrix product of two matrices is a new matrix consisting of the dot product of each row of the first matrix with each column of the second matrix. Using your algorithm above, what would the asymptotic runtime be for computing the product of a $p \times q$ matrix with a $q \times r$ matrix?

Graph Theory

5. **(5 points)** Find three nonisomorphic graphs with the degree sequence $(1, 1, 1, 2, 2, 3)$.

6. **(10 points)** A simple graph contains 10 vertices and 16 edges. Explain why there must be a vertex with degree of at least 4.

7. **(6 points)** A simple graph is called *k-regular* if every vertex of the graph has degree k. Explain why there is no 7-regular simple graph on 19 vertices.

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8. **(5 point bonus)** For the Fibonacci numbers defined as in class ($F_0 = 1$, $F_1 = 1$), prove that $f_n^2 = f_{2n-1} + f_{n-2}^2$.

9. **(5 point bonus)** Prove that $f_0 - f_1 + f_2 - f_3 + \cdots \pm f_n = 1 + (-1)^n f_{n-1}$.

10. **(5 point bonus)** Prove that $\sum_{k=m}^{n} \binom{n}{k} S(k, m) = S(n+1, m+1)$, where $S(n, k)$ is the Stirling number of the second kind.

> On two occasions I have been asked — "Pray, Mr. Babbage, if you put into the machine wrong figures, will the right answers come out?" In one case a member of the Upper, and in the other a member of the Lower House put this question. I am not able rightly to apprehend the kind of confusion of ideas that could provoke such a question. —Charles Babbage

 due March 29, 2013