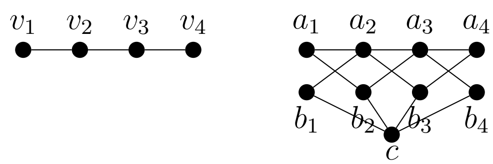
Graphs

1. **(10 points)** Given that every vertex of a finite simple graph G has degree of 2 or more, explain why it must be the case that G contains a cycle.
2. **(5 point bonus)** Let G be a simple graph which contains a cycle C, and a path of length k between two vertices of C. Show that G must contain a cycle of length at least $\lceil\sqrt{k}\rceil$.
3. **(10 points)** Describe how you could construct a bipartite graph on $2n$ vertices, and an ordering of those vertices, such that a greedy coloring would require 4 colors. Show that your example can actually be colored with 2 colors.
4. **(5 point bonus)** For G with n vertices $v_1, v_2, \ldots, v_n$, let $M(G)$ be a graph with $2n+1$ vertices $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n, c$ such that, for v_i adjacent to v_j, it is the case that a_i is adjacent to a_j and b_j, and a_j is adjacent to b_i; furthermore, c is adjacent to every b_i. For example, below is an illustration of P_4 and $M(P_4)$:



Show that $\chi(M(G))$ as always exactly equal to $\chi(G) + 1$.

Algorithms on Graphs

5. **(10 points)** Find a digraph and a flow thereon which could not be improved by naïve flow expansion (i.e. simply attempting to add flow along some path), but which has value only one-third (or less) of the maximum possible flow.

Counting Subject to Symmetry

6. **(10 points)** You are building circular bracelets with 6 beads on them; you have beads in red, yellow, and green. You want to have at least one bead of each color on every bracelet, and two bracelets are considered to be identical if one can be produced by flipping or rotating the other. How many different bracelets are possible?
7. **(10 points)** If p is a prime number, how many distinct circular bracelets are there which can be made using n different colors of beads, if there are no restrictions on which bead colors can be used, and two bracelets are considered to be identical if one can be produced by flipping or rotating the other.
8. **(5 point bonus)** The faces of a cube are to be painted red, blue, and green; each color can be used as many times as desired or not at all. Two cube-paintings are considered to be identical if one is a rotation of the other. How many different ways are there to paint the cubes? Do not brute-force this problem!

> At the other end of the spectrum is, for example, graph theory, where the basic object, a graph, can be immediately comprehended. One will not get anywhere in graph theory by sitting in an armchair and trying to understand graphs better. Neither is it particularly necessary to read much of the literature before tackling a problem: it is of course helpful to be aware of some of the most important techniques, but the interesting problems tend to be open precisely because the established techniques cannot easily be applied. —W.T. Gowers

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