1. **(25 points)** *You are asked to assign your six subordinates (Alice, Bob, Carla, Dave, Ed, and Fiona) to 3 specific projects (codenamed Runcible, Screaming Fist, and Valis).*

   (a) **(5 points)** *How many ways are there to do this if you can assign people freely?*

   An assignation of the subordinates can be broken into 6 subtasks consisting of the assignment of each individual subordinate. We could assign Alice to any of 3 projects, Bob to any of 3, Carla to each of 3, and so forth, for a total of $3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 = 3^6 = 729$ possible assignments.

   (b) **(10 points)** *How many ways are there to do this if you can assign people freely, as long as Carla and Fiona are not assigned to the same project?*

   There are two effective approaches to this problem (and possibly more). We could take the result from part (a), and subtract out those projects in which Carla and Fiona are working on the same project. We can count these by assigning each of Alice, Bob, Dave, and Ed to a project in one of 3 ways each, and then assigning the Carla-Fiona team to a project in one of 3 ways, for a total of 3^5 assignments in which Carla and Fiona are on the same team; subtracting these invalid assignments from the 3^6 total assignments gives $3^6 - 3^5 = 486$.

   Alternatively, you could assign the first five subordinates freely, and then for Fiona, be required to choose a project Carla is *not* on: regardless of which project Carla is on, Fiona will be left with 2 options, so we have $3^5 \cdot 2 = 486$ possible assignments

   (c) **(10 points)** *How many ways are there to do this if each project must receive at least one worker (but there is now no restriction on placing Carla and Fiona on the same job)?*

   This is a canonical example of surjective mapping, which, as we have seen in class, is well-described with an inclusion-exclusion technique. We might let X be the set of all assignments, and then exclude therefrom the sets A in which the Runcible project is empty, B in which the Screaming Fist project is empty, and C in which Valis is empty. We saw in part (a) that $|X| = 3^6$. If one project is empty, the other two are the only viable choices for each assignment, so $|A| = |B| = |C| = 2^6$. Likewise, if any two projects are empty, the remaining choice is the only one possible for an assignment, so $|A \cap B| = |A \cap C| = |B \cap C| = 1^6$. It is impossible for all three projects to be empty, so $|A \cap B \cap C| = 0$. Thus, using inclusion-exclusion, we can find the set of surjective mappings to be

   $$|X - A - B - C| = 3^6 - 3 \cdot 2^6 + 3 \cdot 1^6 - 0 = 540$$

2. **(10 points)** *For the purposes of this question, the English language contains 5 vowels and 21 consonants; also, we call a string of letters a "word" even if it is nonsensical, like the five-letter word "QREFG". How many 6-letter words are there in which exactly 3 letters are vowels?*

   We may build such a six-letter word by performing 3 subprocesses in sequence: first, we pick 3 of the 6 positions to contain vowels, in any of $\binom{6}{3}$ ways. Then, in order, we pick a specific vowel to inhabit each of these spaces, in any of 5^3 ways. Finally, we pick a consonant for each of the remaining 3 spaces, in any of 21^3 ways, for a total of $\binom{6}{3}5^3 21^3 = 23152500$ different words.

3. **(10 points)** *Let five points with integer coordinates be selected in the coordinate plane. Use a pigeonhole argument to show that some two of these five points must have a midpoint which*

has integer coordinates (hint: what property of two points guarantees that their midpoint has integer coordinates?).

Note that the midpoint of (x_1, y_1) and (x_2, y_2) is $(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2})$; in order for this to have integer coordinates, the sums $x_1 + x_2$ and $y_1 + y_2$ must be even. A sum of two integers is even if the two integers have the same parity, so we want to find two points which have the same parity combinations in their coordinates. There are in fact four such combinations: the x-coordinate could be even or odd, and the y-coordinate could be even or odd. We can thus categorize each point with integer coordinates into one of these four "parity classes", and among any five such points, by the pigeonhole principle, two will be categorized into the same class, and will have a midpoint with integer coordinates.

4. **(25 points)** *The Wichway Catering Company provides five different types of sandwiches to your organization, but has some peculiar rules about how many sandwiches of each type can be in an order. They will provide any number of turkey sandwiches and pastrami sandwiches, but insist that every order must contain at least 10 vegetarian sandwiches and no more than 8 roast beef. Finally, an order can only have between 5 and 15 ham sandwiches. So, for instance, an order might consist of no turkey, 15 pastrami, 12 vegetarian, 1 roast beef, and 8 ham (which would be 36 sandwiches in total).*

   (a) **(10 points)** *Letting a_n represent the number of different possible ways to order n sandwiches, find a formula for the ordinary generating function $\sum_{n=0}^{\infty} a_n z^n$.*

   The selections for turkey and pastrami are free, so each of these is characterized by the generating function $1 + z + z^2 = z^3 + \cdots = \frac{1}{1-z}$. The selection of vegetarian sandwiches starts with the term of exponent 10, yielding $z^{10} + z^{11} + z^{12} + \cdots = \frac{z^{10}}{1-z}$. The selection of roast beef is the finite generating function $1 + z + z^2 + \cdots + z^8 = \frac{1-z^9}{1-z}$, and likewise the selection of ham is $z^5 + z^6 + z^7 + \cdots + z^{15} = \frac{z^5 - z^{16}}{1-z}$. Putting all of these together, we see that the generating function for the sequence of ways to get sandwiches of these 5 types collectively is:

$$\frac{z^{10}(1-z^9)(z^5-z^{16})}{(1-z)^5} = \frac{z^{15} - z^{24} - z^{26} + z^{35}}{(1-z)^5}$$

   (b) **(5 points)** *What is the lowest-degree non-zero term in the power series of the generating function you determined above? What is the significance of this term?*

   The power series for this generating function would in fact have zero terms up to z^{15}; this signifies that the restrictions on this particular process are such that ordering fewer than 15 sandwiches is impossible, and the term z^{15} would have coefficient 1 because there is in fact only one way to make the minimum order (10 vegetarian and 5 ham).

   (c) **(10 points)** *Either using your generating function or by other means, determine how many different possible ways there are to place an order for 100 sandwiches.*

   Solving this via generating functions, we may attempt to determine the coefficient of z^{100} in the generating function determined above:

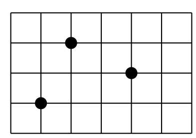
$$\frac{z^{15} - z^{24} - z^{26} + z^{35}}{(1-z)^5} = (z^{15} - z^{24} - z^{26} + z^{35}) \sum_{n=0}^{\infty} \binom{n+4}{4} z^n$$

   And in this summation, we can determine which products of the fixed powers of z in the first factor and the powers of z in the power series have total exponenet 10). We may take, for instance, $z^{15} \cdot \binom{85+4}{4} z^{85} = \binom{89}{4} z^{100}$, and likewise we will get associated with the other terms in the first factor the products $-z^{24} \cdot \binom{76+4}{4} z^{76} = -\binom{80}{4} z^{100}$, $-z^{26} \cdot \binom{74+4}{4} z^{74} = -\binom{78}{4} z^{100}$, and $z^{35} \cdot \binom{65+4}{4} z^{65} = -\binom{69}{4} z^{100}$, for a total z^{100} coefficient of $\binom{89}{4} - \binom{80}{4} - \binom{78}{4} + \binom{69}{4} = 298122$.

Alternatively, one could solve this with inclusion-exclusion. We would start by pre-emptively assigning 10 vegetarian sandwiches and 5 ham, leaving 85 of our 100 left to be assigned. We then let X be the set of all such assignments; with a balls-and-wall paradigm we may assert that $|X| = \binom{85+4}{4} = \binom{89}{4}$. Now, from this, we wish to exclude the set A of all distributions with more than 8 roast beef, and the set B of all distributions with more than 15 ham (i.e., more than 10 *additional* ham, as we have already allocated 5 prior to defining X). To find $|A|$, we would pre-emptively assign 9 roast beef sandwiches, forcing violation of the roast-beef condition, and assign the remaining 76 sandwiches in $\binom{76+4}{4} = \binom{80}{4}$ ways; likewise, we can find $|B|$ by pre-emptively assigning 11 more ham sandwiches (bringing the total to 16), and assign the remaining 74 sandwiches in $\binom{74+4}{4} = \binom{78}{4}$ ways. Finally, $|A \cap B|$ is found by preemptively performing both assignments, which, togethe rwith our initial assignment of 15 sandwiches, leaves only 65 sandwiches left to select, in $\binom{65+4}{4} = \binom{69}{4}$ ways. Invoking inclusion-exclusion, we find that

$$|X - A - B| = \binom{89}{4} - \binom{80}{4} - \binom{78}{4} + \binom{69}{4} = 298122.$$

5. **(10 points)** *How many direct paths are there from the lower left corner to the upper right corner of the following grid which do not pass through any two of the three marked points?*



The total number of gridwalks is enumerated by the set of lists of instructions consisting of four "up" and six "right" steps; such lists can be enumerated by $\binom{10}{4}$, since you must choose four of the ten instructions in sequence to be "up".

From this number we wish to subtract those gridwalks passing through two of the points. Note that it is impossible to pass through both $(2,3)$ and $(4,2)$, since that would require downwards steps; we thus must only concern ourself with walks through $(1,1)$ and $(2,3)$, and walks through $(1,1)$ and $(4,2)$. The former can be broken into three subwalks: a walk from $(0,0)$ to $(1,1)$ in any of $\binom{2}{1}$ ways, a walk from $(1,1)$ to $(2,3)$ in any of $\binom{3}{1}$ ways, and a walk from $(2,3)$ to $(6,4)$ in any of $\binom{5}{1}$ ways; thus this first family of invalid walks has $\binom{2}{1}\binom{3}{1}\binom{5}{1}$ members. Likewise, walks through $(1,1)$ and $(4,2)$ can be broken into the three subwalks: a walk from $(0,0)$ to $(1,1)$ in any of $\binom{2}{1}$ ways, a walk from $(1,1)$ to $(4,2)$ in any of $\binom{4}{1}$ ways, and a walk from $(4,2)$ to $(6,4)$ in any of $\binom{4}{2}$ ways; thus this family of invalid walks has $\binom{2}{1}\binom{4}{1}\binom{4}{2}$ members, so the total is:

$$\binom{10}{4} - \binom{2}{1}\binom{3}{1}\binom{5}{1} - \binom{2}{1}\binom{4}{1}\binom{4}{2} = 132$$

6. **(5 points)** *Find the coefficient of x^2 in the expansion of the polynomial $(2x-4)^9$.*

By the binomial theorem, $(2x-4)^9 = \sum_{i=0}^{9} \binom{9}{i} (2x)^i (-4)^{9-i}$, so the x^2 term specifically will be $\binom{9}{2}(2x)^2(-4)^7$, with a coefficient of $\binom{9}{2} 2^2 (-4)^7 = -2359296$.

7. **(15 points)** *Find the following generating functions:*

   (a) **(5 points)** *Let a_n be the number of ways to place n* distinct *objects in 4 boxes so that each box contains at least 2 items. Determine a formula for the exponential generating function $\sum_{n=0}^{\infty} a_n \frac{z^n}{n!}$.*

   Each box has an associated exponential generating function consisting of one way to distribute 2 or more objects, namely, $\frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots = (e^z - 1 - z)$. Multiplying all four together gives $(e^z - 1 - z)^4$.

   (b) **(10 points)** *Let b_n be the number of ways to write n as a sum of (not necessarily distinct) powers of 2 (e.g. 1, 2, 4, 8, 16, etc.). Determine a formula for the ordinary generating function $\sum_{n=0}^{\infty} b_n z^n$.*

   We may include any number from zero to arbitrarily many "1"s in our sum, each contributing 1 to the total, so the selection of how many "1"s to include in any given partition is given by the generating function $1 + z + z^2 + z^3 + z^4 + \cdots = \frac{1}{1-z}$. Likewise, we may include any number of "2"s, but each contributes 2 to the total, inducing generating function $1 + z^2 + z^4 + z^6 + x^8 + \cdots = \frac{1}{1-z^2}$. Proceeding in this manner, the selection functions for each 2^n will be $\frac{1}{1-z^{(2^n)}}$, so multiplied together we get the generating function:

$$\frac{1}{(1-z)(1-z^2)(1-z^4)(1-z^8)(1-z^{16})\cdots}$$

8. **(5 point bonus)** *Show that $\sum_{i=k}^{n} \binom{i}{k}$ and $\binom{n+1}{k+1}$ count the same objects and are thus equal.*

   The binomial coefficient $\binom{n+1}{k+1}$ clearly counts the $(k+1)$-element subsets of $\{1, 2, \ldots, n+1\}$. We shall show that the sum $\sum_{i=k}^{n} \binom{i}{k}$ counts the same thing.

   Given a particular $(k+1)$-element subset of $\{1, 2, \ldots, n+1\}$, we might be able to characterize it by the value of its largest element, which could plausibly be anything from $k+1$ to $n+1$; let's consider how many such sets there are with a particular largest element $i+1$. Then the remaining k elements of the subset we are looking at would be drawn from $\{1, 2, \ldots, i\}$, which can be done in $\binom{i}{k}$ ways. Summing over the entire range of possible values of i, we get the total of $\sum_{i=k}^{n} \binom{i}{k}$ possible $(k+1)$-element subsets of $\{1, 2, \ldots, n+1\}$.

שתי אבנים בונות שני בתים: שלש אבנים בונות ששה בתים: ארבע אבנים ב ונות ארבעה ועשרים בתים: חמש אבנים בונות מאה ועשרים בתים: שש אבנים בונות ש בע מאות ועשרים בתים: שבע אבנים בונות חמשת אלפים וארבעים בתים: מכאן ו אילך צא וחשוב מה שאין הפה יכול לדבר ואין האוזן יכולה לשמוע

—ספר היצירה פרק ד׳ משנה ט״ז

[Two stones (or letters) build two houses (or words), three stones build six houses, four stones build twenty-four houses, five stones build one hundred twenty houses, six stones build seven and twenty houses, seven stones build five thousand forty houses; thenceforth are numbers which the mouth can not speak and the ear can not hear.]

—Sefer Yetzira, Chapter 4, Verse 16