

For full credit show all of your work (legibly!), unless otherwise specified. **Answers need not (and probably should not) be completely reduced** unless otherwise stated, and may be left in terms of sums, differences, products, quotients, exponentials, factorials, and binomial coefficients.

1. **(25 points)** You are asked to assign your six subordinates (Alice, Bob, Carla, Dave, Ed, and Fiona) to 3 specific projects (codenamed Runcible, Screaming Fist, and Valis).
 - (a) **(5 points)** How many ways are there to do this if you can assign people freely?

 - (b) **(10 points)** How many ways are there to do this if you can assign people freely, as long as Carla and Fiona are not assigned to the same project?

 - (c) **(10 points)** How many ways are there to do this if each project must receive at least one worker (but there is now no restriction on placing Carla and Fiona on the same job)?

2. **(10 points)** For the purposes of this question, the English language contains 5 vowels and 21 consonants; also, we call a string of letters a “word” even if it is nonsensical, like the five-letter word “QREFG”. How many 6-letter words are there in which exactly 3 letters are vowels?

3. **(10 points)** Let five points with integer coordinates be selected in the coordinate plane. Use a pigeonhole argument to show that some two of these five points must have a midpoint which has integer coordinates (hint: what property of two points guarantees that their midpoint has integer coordinates?).

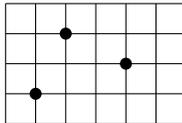
4. **(25 points)** The Wichway Catering Company provides five different types of sandwiches to your organization, but has some peculiar rules about how many sandwiches of each type can be in an order. They will provide any number of turkey sandwiches and pastrami sandwiches, but insist that every order must contain at least 10 vegetarian sandwiches and no more than 8 roast beef. Finally, an order can only have between 5 and 15 ham sandwiches. So, for instance, an order might consist of no turkey, 15 pastrami, 12 vegetarian, 1 roast beef, and 8 ham (which would be 36 sandwiches in total).

(a) **(10 points)** Letting a_n represent the number of different possible ways to order n sandwiches, find a formula for the ordinary generating function $\sum_{n=0}^{\infty} a_n z^n$.

(b) **(5 points)** What is the lowest-degree non-zero term in the power series of the generating function you determined above? What is the significance of this term?

(c) **(10 points)** Either using your generating function or by other means, determine how many different possible ways there are to place an order for 100 sandwiches.

5. **(10 points)** How many direct paths are there from the lower left corner to the upper right corner of the following grid which *do not pass through any two of the three marked points*?



6. **(5 points)** Find the coefficient of x^2 in the expansion of the polynomial $(2x - 4)^9$.
7. **(15 points)** Find the following generating functions:
- (a) **(5 points)** Let a_n be the number of ways to place n *distinct* objects in 4 boxes so that each box contains at least 2 items. Determine a formula for the exponential generating function $\sum_{n=0}^{\infty} a_n \frac{z^n}{n!}$.
- (b) **(10 points)** Let b_n be the number of ways to write n as a sum of (not necessarily distinct) powers of 2 (e.g. 1, 2, 4, 8, 16, etc.). Determine a formula for the ordinary generating function $\sum_{n=0}^{\infty} b_n z^n$.
8. **(5 point bonus)** On the back of this sheet, show that $\sum_{i=k}^n \binom{i}{k}$ and $\binom{n+1}{k+1}$ count the same objects and are thus equal.