1. **(12 points)**

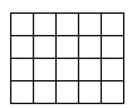
   (a) **(3 points)** *How many even four-digit numbers have at least one 7 appearing as a digit?*
   It is easier to count the total number of even four digit numbers, and subtract those which have no 7s in. There are $9 \cdot 10 \cdot 10 \cdot 5 = 4500$ even four-digit numbers, since there are free choices of the middle two digits, choice of a nonzero digit for the first digit, and choice of an even digit for the last. If we eliminate 7s, each of these decisions loses one choice except the last (which could not have been 7 to begin with) so there are $8 \cdot 9 \cdot 9 \cdot 5$ even four-digit numbers without 7 as a digit. Subtracting these, we get $4500 - 8 \cdot 9 \cdot 9 \cdot 5$ (which is 1260, but need not be explicitly computed).

   (b) **(3 points)** *How many even four-digit numbers have at least one 4 appearing as a digit?*
   As above, we count the total number of even four digit numbers, and subtract those which have no 7s in. The only difference from above is that 4, unlike 7, was a prospective choice for the last digit, so 4-avoidance can be achieved in $8 \cdot 9 \cdot 9 \cdot 4$ ways, giving $4500 - 8 \cdot 9 \cdot 9 \cdot 4$ (which is 1908, but need not be explicitly computed).

   (c) **(6 points)** *How many even four-digit numbers have a 7 or a 4 (or both) appearing as digits?*
   The opposite of this condition is having neither a 7 nor a 4, which can be achieved in $7 \cdot 8 \cdot 8 \cdot 4$ ways. Subtracting from the total pool, we get $4500 - 7 \cdot 8 \cdot 8 \cdot 4 = 2708$ ways to get either a 7 or a 4.

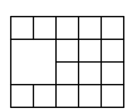
2. **(12 points)**

   (a) **(4 points)** *How many direct paths are there through the following two-dimensional grid?*



   A direct path from one corner to the other would require walking 9 blocks, 4 of them upwards and 5 of them rightwards. Considering a walk as equivalent to the instructions yielding it, we see that a walk may be constructed by selecting 4 positions from a 9-letter string, and marking them "U" (for "up") while the rest are marked "R" for "right". We can do so in $\binom{9}{4}$ ways.

   (b) **(8 points)** *How many direct paths are there through the following two-dimensional grid?*



   This is much like the grid above, but with the caveat that the point with coördinates $(1,2)$ must be avoided. We enumerate $(1,2)$ avoiding walks by subtracting those walks which pass through $(1,2)$ from the pool of $\binom{9}{4}$ walks found above. To count the walks from $(0,0)$ to $(5,4)$ by way of $(1,2)$, we multiply the number of walks from $(0,0)$ to $(1,2)$ by the number of walks from $(1,2)$ to $(5,4)$. These can be seen to be respectively $\binom{3}{1}$ and $\binom{6}{2}$, so there are $\binom{3}{1}\binom{6}{2}$ paths from $(0,0)$ to $(5,4)$ passing through $(1,2)$; thus, the number of $(1,2)$-avoiding paths is $\binom{9}{4} - \binom{3}{1}\binom{6}{2} = 81$.

3. **(12 points)**

(a) **(9 points)** *Prove by induction that $2^n < (n-1)!$ for $n \geq 6$.*

For the base case, note $2^6 = 64$ and $(6-1)! = 120$, so this proposition is true for $n = 6$. Now, for the inductive step, given $2^n < (n-1)!$, we seek to prove that $2^{n+1} < n!$. We do so by observing that $2^{n+1} = 2 \cdot 2^n$, and that $n! = n(n-1)!$. Since $2^n < (n-1)!$ by our inductive hypothesis, and we know that $2 < n$, it follows that $2^{n+1} = 2 \cdot 2^n < n \cot(n-1)! = n!$, demonstrating our inductive step.

(b) **(3 points)** *Evaluate $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{67\cdot 68}$.*

Noting that $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$, the above sum expands into

$$\left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{67} - \frac{1}{68}\right)$$

so all of the terms except the first any last cancel, leaving $1 - \frac{1}{68} = \frac{67}{68}$.

4. **(12 points)**

(a) **(4 points)** *What is the coefficient of x^5yz^2 in the expansion of $(2x + 3y + 4z)^8$?*

By the multinomial theorem, the expansion of $(2x+3y+4z)^8$ contains the term $(2x)^5(3y)^1(4z)^2$ in various permutations $\binom{8}{5,1,2} = \frac{8!}{5!1!2!} = 168$ times; thus, when terms are collected, we will have the term $168(2x)^5(3y)^1(4z)^2 = 168 \cdot 2^5 \cdot 3 \cdot 4^2x^5yz^2$, so the aforementioned coefficient is $168 \cdot 32 \cdot 3 \cdot 16$.

(b) **(8 points)** *Assuming n is odd, evaluate*

$$\binom{n+1}{1} + \binom{n+1}{3} + \binom{n+1}{5} + \binom{n+1}{7} + \cdots + \binom{n+1}{n}$$

Using the binomial property $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$, the above becomes

$$\left[\binom{n}{0} + \binom{n}{1}\right] + \left[\binom{n}{2} + \binom{n}{3}\right] + \left[\binom{n}{4} + \binom{n}{5}\right] + \cdots + \left[\binom{n}{n-1} + \binom{n}{n}\right]$$

which in $\sum_{k=0}^{n} \binom{n}{k}$, known to sum to 2^n.

5. **(12 points)**

(a) **(4 points)** *Agamemnon, Brunhilde, Cihuacoatl, and Daikoku are dividing up a pot containing 8 identical gold coins. In how many ways can they do so?*

We have four individuals whose quantity of the treasure can be respectively numbered x_1, x_2, x_3, and x_4; thus, a division of the pots is exactly a partition $x_1 + x_2 + x_3 + x_4 = 8$. This enumerative statistic can be represented by a filling of 11 spaces with 8 coins and 3 dividers, so that x_1 is equal to the number of coins before the first divider, x_2 the number between the first and second, and so forth. Such an arrangement of coins and dividers can be effected in any of $\binom{11}{3} = 165$ ways.

(b) **(4 points)** *How many anagrams (rearrangements of letters) are there for the word "AGAMEMNON"? The anagrams need not be actual English words.*

We want to construct an arrangement of 2 As, 2 Ns, 2 Ms, a G, an O, and an E. We can do so in a number of ways counted by the multinomial $\binom{9}{2,2,2,1,1,1} = \frac{9!}{2^3}$.

(c) **(4 points)** *How many anagrams of "AGAMEMNON" do not have the "A"s next to each other?*

We counted all the anagrams above; now we may subtract off those which *do* have the As next to each other. We can count this by counting the arrangements of the monolithic element AA, 2 Ns, 2Ms, a G, an O, and an E; this is counted by the multinomial $\binom{8}{1,2,2,1,1,1} = \frac{8!}{2^2}$. We subtract this excluded set from the total number of anagrams to get those which do not have adjacent As: $\frac{9!}{2^3} - \frac{8!}{2^2}$ (which can be, but need not be, algebraically simplified).