1. *You find that you need to buy 22 hats. The hat shop has as many hats as you might desire in four different varieties: stetsons, berets, stovepipes, and pillboxes. Hats within a single variety are identical.*

   (a) *How many different possible ways are there for you to purchase 22 hats?*

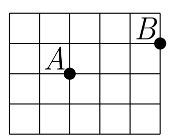
   We may consider an "order" as a placement of some number of unlabeled balls in each of four distinct boxes representing different types of hats (e.g. we could communicate an order of 10 stetsons, 3 berets, 9 stovepipes, and no pillboxes by putting 10 balls in a "stetson" box, 3 in a "beret" box, and 9 in the "stovepipe" box, while the "pillbox" box remains empty). Thus, each purchase can be associated with a distribution of 22 balls among 4 boxes. The count for doing so is a standard enumeration statistic, which we know to be $\binom{22+4-1}{4-1} = \binom{25}{3} = 2300$; alternatively, we could consider the number of ways to place 3 dividers among 22 hats, so that the hats are partitioned into 4 (possibly empty) groups, which will be declared to represent different styles of hat. This would be enumerated with $\binom{22+3}{3}$, as above.

   (b) *Suppose you want to select your lot of 22 hats so that there is at least one hat of each type. How many ways are there to fulfill these instructions?*

   This situation is as in the first part of this problem, except that we constrain each box to contain at least one ball; this would be the standard enumeration statistic $\binom{22-1}{3} = \binom{21}{3} = 1932$.

   Alternatively, the above solution can be justified by pre-emptively assigning one hat of each type, leaving 18 hats left to be assigned.

2. *How many direct paths which pass through at least one of the marked points are there from the lower left corner to the upper right corner of the following grid?*



We shall count the paths through point A: from the lower left corner to point A requires two steps to the right and two steps up, so this section of the path can be walked in any of $\binom{2+2}{2}$ ways. Likewise, traversal from point A to the upper right corner requires three steps to the right and two steps up, so this section of the path can be walked in any of $\binom{3+2}{2}$ ways. Since a walk through point A is built as a concatenation of these two sub-walks, the number of ways to build a path through point A is the product of the number of ways to take each of these partial walks; in other words, $\binom{2+2}{2}\binom{3+2}{2}$.

Likewise, a walk through point B begins with a walk to B, which requires five steps to the right and three steps upwards, which can be ordered in any of $\binom{5+3}{3}$ ways. The walk from B to the end is clearly unique, but a methodological purist might quantify its number as $\binom{1+0}{0}$. Since a walk through B is built from these two subwalks, the nubmer of ways to walk through B is $\binom{5+3}{3}\binom{1+0}{0}$.

However, were we to add these two possibilities, we would in fact double-count those walks which go through both points, and thus we must remove them. Walks through both points can be quantified by considering the number of walks from the beginning to A, which is $\binom{2+2}{2}$ as seen above, the number of walks from A to B, which, since such a walk requires three steps to the right and one up, can be accomplished in $\binom{3+1}{1}$ ways, and the number of walks from B to the end, which as seen above is $\binom{1+0}{0}$. We multiply these quantities to get the number of ways to do all three of these sub-walks in sequence, i.e. to walk from the beginning to the end through A and B.

Thus, the number of walks satisfying the given condition is

$$\binom{4}{2}\binom{5}{2}+\binom{8}{3}\binom{1}{0}-\binom{4}{2}\binom{4}{1}\binom{1}{0}=92.$$

3. *A game is played with a fifty-card deck consisting of cards in the 5 suits of acorns, hearts, leaves, bells, and trumps, numbered 1 to 10. A hand of cards has no intrinsic order.*

   (a) **(5 points)** *How many 5-card hands are there which consist of one card in each suit, with no restrictions on numbers?*

   Our hand has no order, but individual cards may be distinguished by virtue of being different suits. Consider the card in the suit of acorns: it has 10 possible numbers. Likewise, the card in the suit of hearts has 10 possible numbers, as do the cards in leaves, bells, and trumps. A hand is thus uniquely determined by a process of 5 decisions, each of which can be resolved in 10 different ways, so there are $10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 100,000$ possible hands.

   (b) *How many 5-card hands are there with two pairs (two cards in each of two different numbers, and a fifth card in a different number than either)?*

   Let us consider the features of this hand: there are two pairs, which have no intrinsic order, and one leftover card which is distinguished from these two. We must choose two distinct numbers to appear in the pairs, and since there is no intrinsic order, there are $\binom{10}{2}$ ways to do so. The number for the leftover card can be any nubmer which does not appear in the pairs – we will thus have 8 choices for this card. We may thus determine the numbers appearing in our hand any of $\binom{10}{2} \cdot 8$ ways (we could also express this as $\binom{10}{2,1,7}$ to represent choosing from a list of 10 numbers two numbers to designate as pairs, one to designate as a single, and seven to designate as not appearing in this hand).

   Now we must choose suits: Here we can distinguish between the pairs, since the pairs are distinguished by virtue of having distinct numbers chosen (e.g. we could assign suits to the lower-numbered pair, and then the higher-numbered pair). For each pair there are $\binom{5}{2}$ ways to choose suits, and there are $\binom{5}{1}$ ways to choose suits for the singleton. Thus, multiplying the nubmer of ways to choose numbers by the number of ways to choose suits, we get $\binom{10}{2}\binom{8}{1}\binom{5}{2}^2\binom{5}{1} = 180000$.

4. *Let a_1, a_2, a_3, a_4, a_5, and a_6 be integers. Prove that there is a nonempty sum (possibly consisting of a single element) of the form $a_i + a_{i+1} + a_{i+2} + \cdots + a_j$ which is divisible by 6.*

Let us consider the seven values 0, a_1, $a_1 + a_2$, $a_1 + a_2 + a_3$, $\ldots$, $a_1 + a_2 + a_3 + a_4 + a_5 + a_6$ and classify them into the modular congruence classes modulo 6; since there are seven numbers here and six congruence classes, two of them are in the same class, and thus for some $i < j$, it is the case that

$$a_1 + a_2 + a_3 + \cdots + a_{i-1} \equiv a_1 + a_2 + a_3 + \cdots + a_j \pmod 6$$

and thus, subtracting the two sides, it is the case that $a_i + a_{i+1} + a_{i+2} + \cdots + a_j$ is congruent to zero modulo 6 — or in other words, is divisible by 6.

5. *Anna, Béla, Charles, Diane, and Edgar have dug up a treasure chest full of identical gold coins which they will share among themselves. According to their particular piratical code, Anna is to be given at least 10 coins, Béla and Charles are each to get either 5 or 6 coins (they could each receive the same or different numbers), Diane may receive any number of coins, and Edgar must receive at least 4 coins. Let a_n represent the number of ways in which n coins might be distributed.*

   (a) *Find a formula for the ordinary generating function $\sum_{n=0}^{\infty} a_n z^n$.*
   Since any number of coins 10 or greater can be given to Anna (and each number of coins can only be given to her in one way), the function describing the process of providing her with coins is $1z^{10} + 1z^{11} + 1z^{12} + \cdots = \frac{z^{10}}{1-z}$. Likewise, Béla and Charles each can be provided with coins in one of two ways, represented by the polynomial $1z^5 + 1z^6$, Diane's associated function is $1z^0 + 1z^1 + 1z^2 + \cdots = \frac{1}{1-z}$, and Edgar's is $1z^4 + 1z^5 + 1z^6 + \cdots = \frac{z^4}{1-z}$, so the generating function for the distribution as a whole is the product:

$$\frac{z^{10}}{1-z}(z^5+z^6)^2\frac{1}{1-z}\cdot\frac{z^4}{1-z}$$

   (b) *Either using the ordinary generating function or by other means, determine how many ways there are to share a chest of* 30 *coins. You need not arithmetically simplify your answer.*
   We algebraically simplify the generating function above to get

$$\begin{aligned}\sum_{n=0}^{\infty} a_n z^n &= \frac{z^{26}+2z^{25}+z^{24}}{(1-z)^3}\\ &= (z^{26}+2z^{25}+z^{24})\sum_{n=0}^{\infty}\binom{n+2}{2}z^n\end{aligned}$$

   so in order to find a_{30}, we identify the coefficient of z^{30} on the right side; there are three addends contributing towards the z^{30} term, which are $z^{26}\binom{6}{2}z^4$, $2z^{25}\binom{7}{2}z^5$, and $z^{24}\binom{8}{2}z^6$, so that

$$a_{30} = \binom{6}{2} + 2\binom{7}{2} + \binom{8}{2} = 85$$

   Alternatively, we could get the same answer via direct enumerative techniques, but they may be more difficult.

6. *Find the following generating functions:*

   (a) *Let a_n be the number of ways to place n* distinct *objects in 5 boxes so that each box contains fewer than 4 items. Determine the formula for the exponential generating function $\sum_{n=0}^{\infty} a_n \frac{z^n}{n!}$.*
   Each box has one way of containing 0, 1, 2, or 3 items, so each box is associated with the exponential generating function $\frac{1z^0}{0!} + \frac{1z^1}{1!} + \frac{1z^2}{2!} + \frac{1z^3}{3!}$. To get the generating function of the distribution to 5 boxes, we multiply their five associated generating functions, yielding $\left(\frac{1z^0}{0!} + \frac{1z^1}{1!} + \frac{1z^2}{2!} + \frac{1z^3}{3!}\right)^5$.

(b) *Let b_n be the number of ways to partition n into a sum of the numbers 1, 3, and 4. Determine the formula for the ordinary generating function $\sum_{n=0}^{\infty} b_n z^n$.*

We may use no 1s, one 1, two 1s, three 1s, or so forth, each in exactly one way. These choices correspond respectively to contributions of 1, 2, 3, and so forth towards the sum, so the selection of how many 1s to use has associated generating function $1z^0 + 1z^1 + 1z^2 + \cdots = \frac{1}{1-z}$.

Likewise, we may use no 3s, one 3, two 3s, three 3s, or so forth, each in exactly one way. These choices correspond respectively to contributions of 3, 6, 9, and so forth towards the sum, so the selection of how many 3s to use has associated generating function $1z^0 + 1z^3 + 1z^6 + \cdots = \frac{1}{1-z^3}$. A similar argument yields selection function $\frac{1}{1-z^4}$ for the selection of how many 4s to use. Assembling the generating function for the partition by multiplication of the generating functions for the three individual choices involved in creating a partition, we get $\frac{1}{(1-z)(1-z^3)(1-z^4)}$.

7. *Find the number of functions from $\{1, 2, \ldots, 8\}$ to $\{1, 2, 3, 4, 5, 6, 7\}$ so that every even number in the range (i.e. 2, 4, and 6) is mapped to by at least one element of the domain.*

Let X be a set containing every function from $\{1, 2, \ldots, 8\}$ to $\{1, 2, 3, 4, 5, 6, 7\}$; let A_2, A_4, and A_6 be sets respectively containing those functions which do not map any elements of their domains to 2, 4, and 6. Clearly $|X| = 7^8$, since a function can be constructed by mapping each element of the domain to any of 7 different values. Similarly, $|A_2| = |A_4| = |A_6| = 6^8$, since functions lying in any of those sets only have 6 possible values to which each element of the domain can be mapped. Likewise, $|A_2 \cap A_4| = |A_2 \cap A_6| = |A_4 \cap A_6| = 5^8$ and $|A_2 \cap A_4 \cap A_6| = 4^8$.

The functions we wish to enumerate are those not lying in any of A_2, A_4, or A_6, so we specifically wish to find $|X - (A_2 \cup A_4 \cup A_6)|$, and by inclusion-exclusion:

$$\begin{aligned} &|X - (A_2 \cup A_4 \cup A_6)| \\ &\quad = |X| - |A_2| - |A_4| - |A_6| + |A_2 \cap A_4| + |A_2 \cap A_6| + |A_4 \cap A_6| - |A_2 \cap A_4 \cap A_6| \\ &\quad = 7^8 - 3 \cdot 6^8 + 3 \cdot 5^8 - 4^8 = 1832292 \end{aligned}$$