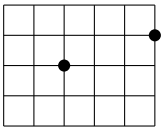
For full credit show all of your work (legibly!), unless otherwise specified. Answers need not (and probably should not) be completely reduced unless otherwise stated, and may be left in terms of sums, differences, products, quotients, exponentials, factorials, and binomial coefficients.

1. You find that you need to buy 22 hats. The hat shop has as many hats as you might desire in four different varieties: stetsons, berets, stovepipes, and pillboxes. Hats within a single variety are identical.

   (a) One example of a hat purchase would be: 10 stetsons, 3 berets, 9 stovepipes, and no pillboxes. How many different possible ways are there for you to purchase 22 hats? (Hint: reframe this with a balls-in-boxes paradigm)

   (b) Suppose you want to select your lot of 22 hats so that there is at least one hat of each type. How many ways are there to fulfill these instructions?

2. How many direct paths which pass through at least one of the marked points are there from the lower left corner to the upper right corner of the following grid?



3. A game is played with a fifty-card deck consisting of cards in the 5 suits of acorns, hearts, leaves, bells, and trumps, numbered 1 to 10. A hand of cards has no intrinsic order. How many 5-card hands are there with two pairs (two cards in each of two different numbers, and a fifth card in a different number than either)?

4. Let $a_1$, $a_2$, $a_3$, $a_4$, $a_5$, and $a_6$ be integers. Prove that there is a nonempty sum (possibly consisting of a single element) of the form $a_i + a_{i+1} + a_{i+2} + \cdots + a_j$ which is divisible by 6. (Hint: consider the seven values 0, $a_1$, $a_1 + a_2$, $a_1 + a_2 + a_3$, $\ldots$, $a_1 + a_2 + a_3 + a_4 + a_5 + a_6$; what can you say about them?)

5. **(15 points)** Anna, Béla, Charles, Diane, and Edgar have dug up a treasure chest full of identical gold coins which they will share among themselves. According to their particular piratical code, Anna is to be given at least 10 coins, Béla and Charles are each to get either 5 or 6 coins (they could each receive the same or different numbers), Diane may receive any number of coins, and Edgar must receive at least 4 coins. Let $a_n$ represent the number of ways in which $n$ coins might be distributed.

   (a) **(5 points)** Find a formula for the ordinary generating function $\sum_{n=0}^{\infty} a_n z^n$.

   (b) **(15 points)** Either using the ordinary generating function or by other means, determine how many ways there are to share a chest of 30 coins. You need not arithmetically simplify your answer.

6. **(10 points)** Find the following generating functions:

   (a) **(5 points)** Let $a_n$ be the number of ways to place $n$ *distinct* objects in 5 boxes so that each box contains fewer than 4 items. Determine the formula for the exponential generating function $\sum_{n=0}^{\infty} a_n \frac{z^n}{n!}$.

   (b) **(5 points)** Let $b_n$ be the number of ways to partition $n$ into a sum of the numbers 1, 3, and 4. Determine the formula for the ordinary generating function $\sum_{n=0}^{\infty} b_n z^n$.

7. Find the number of functions from $\{1, 2, \ldots, 8\}$ to $\{1, 2, 3, 4, 5, 6, 7\}$ so that every even number in the range (i.e. 2, 4, and 6) is mapped to by at least one element of the domain.