1. **(11 points)** *Build a function (or any number of functions, as necessary) to prove that the set of unit fractions* $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$ *and the set of perfect cubes* $\{\ldots, -64, -27, -8, -1, 0, 1, 8, 27, 64 \ldots\}$ *have equal cardinality.*

   We could build a single bijection f by associating 1, $\frac{1}{3}$, $\frac{1}{5}$, and so forth with 0, 1, 8, and so on, and associating $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{6}$, etc. with the negative numbers -1, -8, -27, -64, etc.. To make this association explicit, we could define

$$f(x) = \begin{cases} -\dfrac{1}{8x^3} \text{ if } \dfrac{1}{x} \text{ is even} \\ \left(\dfrac{1}{2x} - \dfrac{1}{2}\right)^3 \text{ if } \dfrac{1}{x} \text{ is odd} \end{cases}$$

   It is probably possible to use a pair of injections, or a pair of surjections, or one of each together with the Cantor-Schröder-Bernstein Theorem to prove this result, but in this case building a single bijection is simpler.

2. **(20 points)** *Let* R *be a relation on* $\mathcal{P}(\{1, 2, 3, 4, 5\})$ *such that* $S\ R\ T$ *iff* S *and* T *have the same number of elements. Prove that* R *is an equivalence relation, and describe the equivalence classes* $[\emptyset]$ *and* $[\{1, 2, 4, 5\}]$.

   To prove reflexivity, note that $|A| = |A|$ is invariably true for any set A, so for any $A \subseteq \{1, 2, 3, 4, 5\}$, $A\ R\ A$.

   TO prove symmetry, let us consider A and B such that $A\ R\ B$. Thus $|A| = |B|$. Since equality is transitive, $|B| = |A|$ and so $B\ R\ A$.

   To prove transitivity, suppose $A\ R\ B$ and $B\ R\ C$. Thus, since $A\ R\ B$, $|A| = |B|$; since $B\ R\ C$, $|B| = |C|$. Thus $|A| = |B| = |C|$ and so $|A| = |C|$, which means $A\ R\ C$.

   The equivalence class of any A is the set of all subsets of $\{1, 2, 3, 4, 5\}$ with the same number of elements as A. So $\emptyset$ is in a equivalence class with every zero-element subset of $\{1, 2, 3, 4, 5\}$, of which there is actually only one:

$$[\emptyset] = \{\emptyset\}$$

   while $\{1, 2, 4, 5\}$ is in a equivalence class with every 4-element subset of $\{1, 2, 3, 4, 5\}$, of which there are five:

$$[\{1, 2, 4, 5\}] = \big\{\{1, 2, 3, 4\}, \{1, 2, 3, 5\}, \{1, 2, 4, 5\}, \{1, 3, 4, 5\}, \{2, 3, 4, 5\}\big\}.$$

3. **(15 points)** *Let functions* $f : A \to B$, $g : B \to C$, *and* $h : C \to A$ *be such that for every* $a \in A$, $h(g(f(a))) = a$. *Prove that* f *must be injective.*

   Proof. Suppose $f(a) = f(a')$. Then applying $h \circ g$ to both sides yields that $h(g(f(a))) = h(g(f(a')))$; from the given relationship between h, g, and f, the left and right sides of this equation are equal to a and a' respectively, so $a = a'$. □

4. **(24 points)** *For each of the following functions, identify whether it is injective, surjective, neither, or both. Briefly justify your claim.*

   (a) **(8 points)** $f : \mathbb{N} \to \mathcal{P}(\mathbb{N}) \to$ *given by the rule that* $f(n)$ *is equal to the set of all multiples of* n.

This function is injective: if $f(a) = f(a')$, then $f(a)$ and $f(a')$ must have the same least element. But the smallest multiple of a is a, and the smallest multiple of a' is a', and so $a = a'$.

This function is not surjective: No finite sets are in its range, and neither are a great many infinite sets which are not sets of multiples.

(b) **(8 points)** *$g : \mathbb{Q} \to \mathbb{Z}$ given by letting $g(x)$ be the greatest integer less than or equal to x; that is to say, the result of "rounding down" from x to an integer.*

This function is not injective: $g(\frac{1}{2}) = g(\frac{1}{3}) = 0$.

This function is surjective: For every $n \in \mathbb{Z}$, $g(n) = n$; if we wanted to be more exotic, we might note that, for instance, $g(n + \frac{1}{2}) = n$ as well.

(c) **(8 points)** *$g : \mathbb{Z} \to \mathbb{Z}$ given by the formula $g(n) = n^4 + 3$.*

This function is not injective: $g(1) = g(-1) = 4$.

This function is not surjective: there is no integer n such that $g(n) = 0$, for instance, because n^4 is never equal to -3.

5. **(15 points)** *By crafting an appropriate function, prove that for any (not necessarily finite) set S, $|S| \leq |S \times S|$.*

A simple function to build is an injection $f : S \to S \times S$. An easy way to build such an injection is to let $f(x) = (x, x)$: this is not remotely a surjection, but it will be injective, since if $f(x) = f(y)$, then $(x, x) = (y, y)$, which is the case only when $x = y$.

Alternatively, we could build a surjection $g : S \times S \to S$. One such example is $g((x, y)) = x$. This is clearly a surjection since for any $x \in S$, if we chose any old element $c \in S$, $g((x, c))$ would equal x and thus whatever x we are considering is in fact in the range of g.

6. **(15 points)** *For a function $f : A \to B$ and a set $S \subseteq A$, let $f_S : S \to B$ be given by the rule that for every $a \in S$, $f_S(a) = f(a)$ (note that since $S \subseteq A$, this is always well-defined). Prove that if f_S is surjective, then f is also surjective.*

Proof. Consider an arbitrary $b \in B$. By surjectivity of f_S, there is some $a \in S$ such that $f_S(a) = b$. Then by the definition of f_S, $f(a) = f_S(a) = b$. Thus, since for an arbitrary $b \in B$, there is an $a \in A$ (specirfically within S, but $S \subseteq A$) such that $f(a) = b$, it follows that f is by definition surjective. □