1. **(15 points)** *Write out truth tables for each of the following statements (you may write them all in one truth table, if you wish):*

   (a) $(\sim P) \vee (Q \to P)$.

P	Q	$\sim P$	$Q \to P$	$(\sim P) \vee (Q \to P)$
T	T	F	T	T
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

   (b) $P \to (P \wedge Q)$.

P	Q	$P \wedge Q$	$P \to (P \wedge Q)$
T	T	T	T
T	F	F	F
F	T	F	T
F	F	F	T

   (c) $(P \vee Q) \leftrightarrow (P \to \sim Q)$.

P	Q	$P \vee Q$	$\sim Q$	$P \to \sim Q$	$(P \vee Q) \leftrightarrow (P \to \sim Q)$
T	T	T	F	F	F
T	F	T	T	T	T
F	T	T	F	T	T
F	F	F	T	T	F

2. **(8 points)** *Write the negation of the true statement "For every integer n, n^3 is also an integer" as a quantified statement (using a universal or existential quantifier, either in words or in symbols).*

   The negation of a universal statement is an existential statement of a negative, so we could restate "it is not the case that for every integer n, n^3 is also an integer" as the existential statement "there is an integer n such that n^3 is not an integer", or, symbolically, $\exists n \in \mathbb{Z} : n^3 \notin \mathbb{Z}$.

3. **(15 points)** *Determine the converse of the false statement "If an integer n is odd, then n is a prime number." Is the converse true? Either briefly justify your statement or provide a counterexample.*

   The converse of this statement is "If n is a prime number, then n is odd.". This is false, since 2 is a prime number which is not odd.

4. **(15 points)** *Let a and b be nonzero integers. Prove that if $a \mid b$ and $b \mid a$, then either $a = b$ or $a = -b$.*

   Proof. Interpreting our premises in terms of the definition of divisibility, we know there are integers k and ℓ such that $b = ka$ and $a = \ell b$. Thus, $b = k(\ell b)$; since b is nonzero we may cancel it from both sides of the equality to conclude that $k\ell = 1$, Since the only integer factors of 1 are 1 and -1, it follows that either $k = \ell = 1$ or $k = \ell = -1$; in the former case, $a = b$, and in the latter, $a = -b$. □

One can alternatively prove the contrapositive, although it's a bit tricky:

Proof. We shall instead prove the contrapositive: if $a \neq b$ and $b \neq a$, then either $a \nmid b$ or $b \nmid a$. Our premise that $a \neq \pm b$ can be more succinctly stated as asserting that $|a| \neq |b|$. Without loss of generality we could specify $|a| > |b|$, and from this we shall seek to show that $a \nmid b$. Noting that $\left|\frac{b}{a}\right| = \frac{|b|}{|a|} < 1$, and that since $b \neq 0$, $\left|\frac{b}{a}\right| > 0$; thus $0 < \left|\frac{b}{a}\right| < 1$, so that $\frac{b}{a}$ is an element of $(-1,0) \cup (0,1)$, which notably contains no integers. Thus there is no integer k such that $b = ka$, so $a \nmid b$. □

5. **(18 points)** *Prove the following two statements:*

   (a) **(6 points)** *For an odd integer b and an integer n, if n is even, then bn is even.*

   Proof. Assuming the premise that n is even, we may define $n = 2k$ for some integer k. Then, $bn = b(2k) = 2(bk)$. Since bk is an integer, it follows that bn, which is twice an integer, is even. □

   Note: b being odd is in fact an unnecessary condition above, as we never use it. We could thus write a stronger version of the same result by simply asserting that b must be some integer.

   (b) **(12 points)** *For an odd integer a and an integer n, if an is even, then n is even.*

   Proof. We shall prove this by appeal to the contrapositive, namely that if n is not even (i.e. odd), then an is not even (i.e. odd). We assume our contrapositive premise, and may thus assert that $n = 2k+1$ for some integer k. In addition, since a is odd, $a = 2\ell + 1$ for some integer ℓ. Then,

$$an = (2\ell+1)(2k+1) = 4k\ell + 2\ell + 2k + 1 = 2(2k\ell + \ell + k) + 1$$

   Since $2k\ell + \ell + k$ is an integer, the above equation demonstrates that an is odd, as desired. □

6. **(15 points)** *Prove by contradiction that, for an irrational number α and a nonzero rational number x, their product αx is irrational.*

Proof. Assume contrariwise that αx is rational. Since x and αx are rational, we may write that $x = \frac{p}{q}$ and $\alpha x = \frac{a}{b}$ for integers a,b,p,q with nonzero q and b. Furthermore, since $x \neq 0$, p is nonzero as well. Then, we may write

$$\frac{a}{b} = \alpha x = \alpha \cdot \frac{p}{q}$$

and since p is nonzero, we may divide both sides of this equation by p and multiply by q to yield:

$$\frac{aq}{bp} = \alpha$$

which contradicts our premise that α is irrational, since it is here expressed as a ratio of integers whose denominator is nonzero (since both b and p are nonzero). □

7. **(14 points)** *Prove or disprove each of the following existential statements:*

(a) *There is a smallest positive real number.*

Disproof. Suppose counterfactually that there is a smallest positive real number ϵ; in other words, for any $x > 0$, it is true that $x \geq \epsilon$. Let us consider specifically $\frac{\epsilon}{2}$: since $\epsilon > 0$, $\frac{\epsilon}{2} > 0$, and thus $\frac{\epsilon}{2} \geq \epsilon$, since we have asserted that ϵ is the smallest positive real number. Dividing both sides by the nonzero number epsilon, we get that $\frac{1}{2} \geq 1$, an obvious falsehood, so our assumption that there was a smallest positive real number is incorrect. □

(b) *If a does not divide b, then there is a number n which is divisible by b but not by a.*

Proof. This is easily demonstrated by example: specifically, we could choose n to be equal to b. Then since $b \mid b$, $n \mid b$, but since $b \nmid a$, $n \nmid a$. □