1. **(17 points)** *For each natural number i, let S_i be the half-open interval $[1, 2+\frac{1}{i})$. Calculate the results of the following indexed set operations:*

   (a) $\bigcap_{i=1}^{3} S_i$.

   $\bigcap_{i=1}^{3} S_i = S_1 \cap S_2 \cap S_3 = [1,3) \cap [1,\frac{5}{2}) \cap [1,\frac{7}{3}) = [1,\frac{7}{3})$.

   (b) $\bigcup_{i=1}^{3} S_i$.

   $\bigcup_{i=1}^{3} S_i = S_1 \cup S_2 \cup S_3 = [1,3) \cup [1,\frac{5}{2}) \cup [1,\frac{7}{3}) = [1,3)$.

   (c) $\bigcap_{i=1}^{\infty} S_i$.

   We may note that for every i, $[1,2] \subseteq [1, 2+\frac{1}{i})$, since $2 < 2+\frac{1}{i}$. However, for any $x > 2$, there is some S_i such that $x \notin S_i$, since $x > 2+\frac{1}{i}$ for some i. Thus the family of numbers which lie in every S_i is exactly $[1,2]$.

   (d) $\bigcup_{i=1}^{\infty} S_i$.

   Note that every S_i is a subset of S_1; the latter sets are smaller and have the same leftmost endpoint. Thus this union, starting with S_1, never grows any larger, so the result of this infinite union is $[1,3)$.

2. **(12 points)** *Prove or disprove: for sets A and B, if $A \cap B = A$, then $A \subseteq B$.*

   Proof. Let $x \in A$. By our premise, $A = A \cap B$, so $x \in A \cap B$. Thus, $x \in B$. Since we have shown an arbitrary element of A is in B, it follows that $A \subseteq B$. □

3. **(18 points)** *Give examples of sets satisfying the following conditions, or explain why they cannot be met:*

   (a) *sets A, B, and C such that $A \subsetneq B$, $A \in C$, and $B \in C$.*
   A minimal example would be $A = \emptyset$, $B = \{1\}$, and $C = \big\{\emptyset, \{1\}\big\}$. A more natural example might be $A = \{1,2,3\}$, $B = \{1,2,3,4\}$, and $C = \big\{\{1,2,3\}, \{1,2,3,4\}, 5\big\}$.

   (b) *sets S and T such that $T = \mathcal{P}(S)$ and $S \cap T \neq \emptyset$.*
   A minimal example would be $S = \{\emptyset\}$, $T = \big\{\emptyset, \{\emptyset\}\big\}$, which would have nonempty intersection $\{\emptyset\}$. A more natural example might be $S = \big\{1, \{1\}\big\}$, $T = \Big\{\emptyset, \{1\}, \big\{\{1\}\big\}, \big\{1, \{1\}\big\}\Big\}$, which have nonempty intersection $\big\{\{1\}\big\}$. In general, S would need to have some set both as a subset and an an element.

   (c) *sets X, Y, and Z such that $X \cup Y = Z$ and $Z \subsetneq X$.*
   This is impossible, because if $Z \subsetneq X$, then every element of Z is in X but not vice versa, so some element of X is not in Z. But if $X \cup Y = Z$, then every element of X is in Z.

4. **(15 points)** *Prove that if a, b, m, and n are integers such that $m \mid n$ and $a \equiv b \pmod{n}$, then $a \equiv b \pmod{m}$.*

Proof. By our premises, $n = km$ and $(a-b) = \ell n$ for some integers k and ℓ. Then $(a-b) = \ell(km) = (\ell k)m$, so $a \equiv b \pmod{m}$. □

5. **(18 points)** *For each of the following relations, determine whether it is reflexive, symmetric, and/or transitive, providing a brief explanation for the properties which hold and a counterexample for properties which do not hold.*

- *The relation R_1 on $\{1,2,3\}$ given by $R_1 = \{(1,1),(1,2),(2,2)\}$.*
  This relation is nonreflexive, since $(3,3) \notin R_1$.
  This relation is nonsymmetic, since $(1,2) \in R_1$ but $(2,1) \notin R_1$.
  This relation is transitive, since the only cases where $(a,b),(b,c) \in R_1$ are when either $a = b = 1$ and $c = 2$ or $a = 1$ and $b = c = 2$, and in either case $(a,c) \in R_1$ as well.
- *The relation R_2 on $\mathbb{R}$ given by $x\ R_2\ y$ iff $|x| \geq |y|$.*
  This relation is reflexive, since for every $x \in \mathbb{R}$, $|x| \geq |x|$.
  This relation is nonsymmetric, since, for example, $|2| \geq |1|$ but $|1| \not\geq |2|$.
  This relation is transitive, since whenever $|x| \geq |y|$ and $|y| \geq |z|$, it is well-known that $|x| \geq |z|$.
- *The relation R_3 on $\mathcal{P}(\mathbb{N})$ given by $A\ R_3\ B$ iff $A \cap B \neq \emptyset$.*
  This relation is nonreflexive, since $\emptyset \cap \emptyset = \emptyset$, so $\emptyset \not R_3\ \emptyset$.
  This relation is symmetric, since $A \cap B = B \cap A$ and thus either both intersections or neither intersection will be empty.
  This relation is nontransitive. For example, $\{1,2\}\ R_3\ \{2,3\}$ and $\{2,3\}\ R_3\ \{3,4\}$, since $\{1,2\} \cap \{2,3\} \neq \emptyset$ and $\{2,3\} \cap \{3,4\} \neq \emptyset$; however $\{1,2\} \not R_3\ \{3,4\}$, since $\{1,2\} \cap \{3,4\} = \emptyset$.

6. **(20 points)** *Prove that for any positive integer n, it is the case that $1(1!) + 2(2!) + \cdots + n(n!) = (n+1)! - 1$. Recall that $k! = k(k-1)(k-2)\cdots(3)(2)(1)$.*

Proof. We proceed by induction; note that the base case $n = 1$ is trivially true, as $1(1!) = 1 = 2! - 1$. Now, we assume, for a specific n, that it is true that

$$1(1!) + 2(2!) + \cdots + n(n!) = (n+1)! - 1$$

and seek to prove that

$$1(1!) + 2(2!) + \cdots + n(n!) + (n+1)(n+1)! = (n+2)! - 1$$

To do so, we may add $(n+1)(n+1)!$ to both sides of our inductive hypothesis, and proceed via arithmetic:

$$\begin{aligned} 1(1!) + 2(2!) + \cdots + n(n!) &= (n+1)! - 1 \\ 1(1!) + 2(2!) + \cdots + n(n!) + (n+1)(n+1)! &= (n+1)! - 1 + (n+1)(n+1)! \\ &= (1 + n + 1)(n+1)! - 1 \\ &= (n+2)(n+1)! - 1 = (n+2)! - 1 \end{aligned}$$

□