1. *Prove that* (1234) *cannot be written as the product of 3-cycles.*

   Each 3-cycle is an even permutation, noting that $(abc) = (ab)(bc)$ is a way of writing (abc) as a product of two (i.e., an even number of) swaps. Thus any product of 3-cycles is also an even permutation. But $(1234) = (12)(23)(34)$, which demonstrates that it is odd and not even.

2. *Find an element of S_9 with order 20, and prove that no element of S_8 has order 20.*

   (1234)(56789) is such an element of S_9, since it is a product of a 4-cycle and a 5-cycle and $\mathrm{lcm}(4,5) = 20$.

   The possible orders in S_8 are those equal to the least common multiples of the elements of a partition of the number 8. There aren't that many of them (22 in total), and it's easy to check them all:

$$
\begin{aligned}
8 &= 8 & \mathrm{lcm}(8) &= 8 \\
&= 7+1 & \mathrm{lcm}(7,1) &= 7 \\
&= 6+2 & \mathrm{lcm}(6,2) &= 6 \\
&= 6+1+1 & \mathrm{lcm}(6,1,1) &= 6 \\
&= 5+3 & \mathrm{lcm}(5,3) &= 15 \\
&= 5+2+1 & \mathrm{lcm}(5,2,1) &= 10 \\
&= 5+1+1+1 & \mathrm{lcm}(5,1,1,1) &= 5 \\
&= 4+4 & \mathrm{lcm}(4,4) &= 4 \\
&= 4+3+1 & \mathrm{lcm}(4,3,1) &= 12 \\
&= 4+2+2 & \mathrm{lcm}(4,2,2) &= 4 \\
&= 4+2+1+1 & \mathrm{lcm}(4,2,1,1) &= 4 \\
&= 4+1+1+1+1 & \mathrm{lcm}(4,1,1,1,1) &= 4 \\
&= 3+3+2 & \mathrm{lcm}(3,3,2) &= 6 \\
&= 3+3+1+1 & \mathrm{lcm}(3,3,1,1) &= 3 \\
&= 3+2+2+1 & \mathrm{lcm}(3,2,2,1) &= 6 \\
&= 3+2+1+1+1 & \mathrm{lcm}(3,2,1,1,1) &= 6 \\
&= 3+1+1+1+1+1 & \mathrm{lcm}(3,1,1,1,1,1) &= 3 \\
&= 2+2+2+2 & \mathrm{lcm}(2,2,2,2) &= 2 \\
&= 2+2+2+1+1 & \mathrm{lcm}(2,2,2,1,1) &= 2 \\
&= 2+2+1+1+1+1 & \mathrm{lcm}(2,2,1,1,1,1) &= 2 \\
&= 2+1+1+1+1+1+1 & \mathrm{lcm}(2,1,1,1,1,1,1) &= 2 \\
&= 1+1+1+1+1+1+1+1 & \mathrm{lcm}(1,1,1,1,1,1,1,1) &= 1
\end{aligned}
$$

   and we note that 20 does not appear on the right side of this list: the elements of S_8 have orders 1, 2, 3, 4, 5, 6, 7, 8, 10, 12, and 15.

3. *Prove that S_4 is not isomorphic to D_{12}.*

   One easy way to demonstrate this is to note that every element of S_4 has order of 4 or less, whereas D_{12} has elements of order 6 and 12. Since they have different orders on their elements, these two groups cannot be isomorphic.

4. *Recall that the* centralizer *of an element a of a group, denoted, $C(a)$, is the subgroup consisting of all b such that $ab = ba$.*

Prove that for any $a, g \in G$, $C(a)$ is isomorphic to $C(gag^{-1})$.

We may attempt to construct an explicit isomorphism. Note that, by the definition of a centralizer, $x \in C(gag^{-1})$ if and only if $gag^{-1}x = xgag^{-1}$. Multiplying this equation on the left by g^{-1} and on the right by g, we find that thisd criterion is identical to

$$a(g^{-1}xg) = (g^{-1}xg)a$$

which is to say, $x \in C(gag^{-1})$ if and only if $g^{-1}xg \in C(a)$. This furnishes a fairly obvioULS bijection between the two groups: namely, we define the function $\varphi : C(gag^{-1}) \to C(a)$ given by $\varphi(x) = g^{-1}xg$. We may demonstrate that this is a bijection by explicitly constructing its inverse $\varphi^{-1}(x) = gxg^{-1}$.

Now, all we need to do is show this is actually a homomorphism, i.e., that $\varphi(xy) = \varphi(x)\varphi(y)$ for any $x, y \in C(gag^{-1})$. This is easily done:

$$\begin{aligned}
\varphi(xy) &= g^{-1}(xy)g \\
&= g^{-1}x(gg^{-1})yg \\
&= (g^{-1}xg)(g^{-1}yg) \\
&= \varphi(x)\varphi(y)
\end{aligned}$$

5. *Describe* $\mathrm{Aut}(D_5)$ *as fully as possible: identify its order, its elements, and any multiplicative rules you can find.*

We might note that an automorphism of D_5 is fully described by the images of r and f, since r and f generate the group. r is an element of order 5 and must be mapped to another element of order 5; f is an element of order 2 and must be mapped to another element of order 2. There are thus four choices of plausible image for r (r itself, r^2, r^3, and r^4) while f could be mapped to any of the five elements of the form fr^i (with i from 0 to 4). We could thus posit the existence of 20 potential isomorphisms, which we might label $\varphi_{1,0}, \varphi_{1,1}, \ldots, \varphi_{4,4}$ with the names indicating that $\varphi_{i,j}(r) = r^i$ and $\varphi_{i,j}(f) = fr^j$.

We may ask, however, if these potential automorphisms actually follow the requirements of a homomorphism. Let us experiment with how they would act on various rotation-and-flip combinations:

$$\varphi_{i,j}(r^a r^b) = \varphi_{i,j}(r^{a+b}) = (r^i)^{a+b} = (r^i)^a (r^i)^b = \varphi_{i,j}(r^a)\varphi_{i,j}(r^b)$$

$$\varphi_{i,j}(r^a f) = \varphi_{i,j}(fr^{-a}) = fr^j (r^i)^{-a} = (r^i)^a fr^j = \varphi_{i,j}(r^a)\varphi_{i,j}(f)$$

$$\varphi_{i,j}(fr^a) = fr^j (r^i)^a = \varphi_{i,j}(f)\varphi_{i,j}(r^a)$$

so each of these 20 mappings is in fact an automorphism, so $\mathrm{Aut}(D_5) = \{\varphi_{10}, \ldots, \varphi_{44}\}$, and $|\mathrm{Aut}(D_5)| = 20$. Note that φ_{10} is the identity automorphism. We can determine pretty straightforward multiplicative rules for these elements by considering the effects of compositions on r and f:

$$\varphi_{i,j}\left(\varphi_{i',j'}(r)\right) = \varphi_{i,j}(r^{i'}) = (r^i)^{i'} = r^{ii'}$$

$$\varphi_{i,j}\left(\varphi_{i',j'}(f)\right) = \varphi_{i,j}(fr^{j'}) = fr^j (r^i)^{j'} = fr^{j+ij'}$$

so our multiplication rule is $\varphi_{i,j}\varphi_{i',j'} = \varphi_{ii',j+ij'}$, which is more than a little messy.