1. *Prove that if N and M are normal subgroups of G, then NM is also a normal subgroup of G.*

   Proof. There are two facts to be proven: that NM is a subgroup of G at all, and that it is normal in G.

   To prove the first, we consider the product of two prototypical elements of NM, n_1m_1 and n_2m_2, where $n_1, n_2 \in N$ and $m_1, m_2 \in M$. This product is $(n_1m_1)(n_2m_2) = n_1(m_1n_2)m_2$. Note that, because $n_2 \in N$ and $m_1 \in G$ and $N \trianglelefteq G$, $m_1n_2m_1^{-1} \in N$; let us denote this expression by n_2' for brevity, so that $m_1n_2 = n_2'm_1$. Then $n_1(m_1n_2)m_2 = n_1n_2'm_1m_2$; since $n_1n_2' \in N$ and $m_1m_2 \in M$, this product is in NM and we have shown that NM is closed under multiplication.

   Likewise, let us consider the inverse of a prototypical element of NM: $(nm)^{-1} = m^{-1}n^{-1}$. As above, we may use normality to note that $n' = m^{-1}n^{-1}m$ is an element of N, so $m^{-1}n^{-1} = n'm^{-1}$, and thus $(nm)^{-1} = n'm^{-1} \in N$, so NM is closed under inversion.

   Finally, we wish to show that NM is itself normal in G, so that for any $g \in G$ and $nm \in NM$, $gnmg^{-1} \in NM$. This is surprisingly easy: note that $gnmg^{-1} = (gng^{-1})(gmg^{-1})$, and that by normality the first term of this product is in N, and the second in M. □

2. *For a group G, let $S = \{x^{-1}y^{-1}xy : x, y \in G\}$ and let H be the subgroup of G generated by S. Prove that H is a* normal *subgroup of G and that G/H is Abelian.*

   Proof. To prove that H is normal in G, let us consider the element of S described by an arbitrary pair $x, y \in G$, and see what the result of conjugation by an arbitrary $g \in G$ is:

   $$g^{-1}(x^{-1}y^{-1}xy)g = (g^{-1}x^{-1}g)(g^{-1}y^{-1}g)(g^{-1}xg)(g^{-1}yg) = (g^{-1}xg)^{-1}(g^{-1}yg)^{-1}(g^{-1}xg)(g^{-1}yg)$$

   which is indeed itself an element of S, defined by the pair $g^{-1}xg, g^{-1}yg$. Now, since S is closed under conjugation, it is easy to see that $H = \langle S \rangle$ is also closed under conjugation, since any element h of H can be written as $s_1s_2s_3 \ldots s_k$, from which $g^{-1}hg$ can be rewritten as

   $$(g^{-1}s_1g)(g^{-1}s_2g)\cdots(g^{-1}s_kg)$$

   which is a product of elements of S, and is thus in H.

   To show that G/H is Abelian, we want to show that $abH = baH$ for any $a, b \in G$. This is pretty easy, since $b^{-1}a^{-1}ba \in H$, and so $ab(b^{-1}a^{-1}ba) \in abH$. However, this expression simplifies to ba, so $ba \in abH$. Clearly $ba \in baH$, and since overlapping cosets must be identical, we know that $abH = baH$. □

3. *Let $G = \{\pm 1, \pm i, \pm j, \pm k\}$ with multiplication rules such that negation works as it normally does, and $i^2 = j^2 = k^2 = -1$, and $ij = k$, $ji = -k$, $jk = i$, $kj = -i$, $ki = j$, and $ik = -j$ (This group is known as the* quaternion group*; it is something of a crossbreed of complex numbers and vector cross products). Show that the quaternion group can* not *be written as an internal direct product of any nontrivial subgroups.*

   This is actually extremely simple! If $G = G_1 \times G_2$, then $|G| = |G_1| \cdot |G_2|$. Here $|G| = 8$, so G_1 and G_2 must have orders which describe a binary factorization of 8; wlog we might assume $|G_1| \geq |G_2|$, in which case the only possibilities are $|G_1| = 8; |G_2| = 1$ and $|G_1| = 4; |G_2| = 2$. The former is trivial (since then $G_1 = G$ and G_2 is the identity-only group); the latter we shall show to be impossible with ease. Note that every group of order 2 or 4 is Abelian; thus, no choice of G_1 and G_2 yields a non-Abelian group $G_1 \times G_2$. Since G is actually non-Abelian, this proves the impossibility of such a decomposition.

4. *Prove (with homomorphisms or directly) that $D_{2n}/\langle r^n \rangle \simeq D_n$.*

Proof. Let φ be a homomorphism from D_{2n} to D_n given by the rather simplistic mapping of r to r and f to f. Note that this is a homomorphism solely because when $r^k = r^\ell$ in D_{2n}, it must be true that $k \equiv \ell \pmod{2n}$, and thus $k \equiv \ell \pmod{n}$ and $r^k = r^\ell$ in D_n as well. This homomorphism is surjective because any element in D_{2n} will be the image of the identically-named element of D_n.

Note that $\ker \varphi = \{e, r^n\} = \langle r^n \rangle$. Thus by the First Isomorphism Theorem, $D_{2n}/\langle r^n \rangle \simeq D_n$. □

5. *Let G be a group such that for certain primes p and q, there is a homomorphism from G* onto *(i.e. surjectively mapping to) Z_{pq}. Prove that G has normal subgroups of index p and q.*

Let φ be the given surjective homomorphism from p onto Z_{pq}, and let π be the homomorphism from Z_{pq} to Z_p given by $\pi(1) = 1$ (so that $\pi(k)$ will be the remainder on dividing k by p). This homomorphism will obviously be surjective, since $Z_p = \{\pi(1), \pi(2), \ldots, \pi(p)\}$, and thus the composition $\pi \circ \varphi$ will be a surjective mapping from G to Z_p, so $G/\ker(\pi \circ \varphi) \cong Z_p$. Thus

$$[G : \ker(\pi \circ \varphi)] = |G/\ker(\pi \circ \varphi)| = p$$

so $\ker(\pi \circ \varphi)$ is a normal subgroup of G of index p. A similar construction yields a normal subgroup of G of order q.