1. *Determine all of the homomorphisms from Z_{20} to itself. For each homomorphism, determine its kernel.*

   Since Z_{20} is cyclic, a homomorphism is uniquely determined by the image of a generator (for simplicity, we'd consider the image of 1). There are thus 20 homomorphisms: we could denote them $\varphi_0, \varphi_1, \ldots, \varphi_1 9$ according to the definition $\varphi_k(1) = k$, which would induce the general rule $\varphi_k(x) = kx$. The kernels will be as follows:

$$\ker \varphi_0 = Z_{20} = \{0, 1, \ldots, 19\}$$
$$\ker \varphi_1 = \ker \varphi_3 = \ker \varphi_7 = \ker \varphi_9 = \ker \varphi_{11} = \ker \varphi_{13} = \ker \varphi_{17} = \ker \varphi_{19} = \{0\}$$
$$\ker \varphi_2 = \ker \varphi_6 = \ker \varphi_{10} = \ker \varphi_{14} = \ker \varphi_{18} = \langle 10 \rangle = \{0, 10\}$$
$$\ker \varphi_4 = \ker \varphi_8 = \ker \varphi_{12} = \ker \varphi_{16} = \langle 5 \rangle = \{0, 5, 10, 15\}$$
$$\ker \varphi_5 = \ker \varphi_{15} = \langle 4 \rangle = \{0, 4, 8, 12, 16\}$$
$$\ker \varphi_{10} = \langle 2 \rangle = \{0, 2, \ldots, 18\}$$

2. *Prove that if $H \trianglelefteq G$ and $K \trianglelefteq G$ and $H \cap K = \{e\}$, then G is isomorphic to a subgroup of $G/H \oplus G/K$.*

   We shall build an injective homomorphism φ from G into $G/H \oplus G/K$; since this is an injection, it can be made bijective by limiting its domain, and will then become an isomorphism, so that $\varphi(G)$ is a subgroup of $G/H \oplus G/K$ which is isomorphic to G.

   Our homomorphism will be defined very naïvely: let $\varphi(g) = (gH, gK)$. It is abundantly clear by the way multiplications of cosets of normal groups and multiplications in direct products are calculated that this will in fact be a homomorphism:

$$\varphi(g_1)\varphi(g_2) = (g_1H, g_1K)(g_2H, g_2K) = (g_1Hg_2H, g_1Kg_2K) = ((g_1g_2)H, (g_1g_2)K) = \varphi(g_1g_2)$$

   so now all that remains is to prove that this homomorphism is injective. Suppose $\varphi(g) = \varphi(g')$. Thus $(gH, gK) = (g'H, g'K)$, so $gH = g'H$ and $gK = g'K$. We may thus conclude, based on our knowledge of when cosets coincide, that $g^{-1}g' \in H$ and $g^{-1}g' \in K$. However, the only thing which is both an element of H and or K is e, so $g^{-1}g' = e$ and thus $g = g'$.

3. *Prove that if G and H are finite groups and $|G|$ and $|H|$ are relatively prime, for any homomorphism $\varphi : G \to H$, $\ker \varphi = G$.*

   We know from the First Isomorphism Theorem (or even some simpler results) that $|\varphi(G)| = \frac{|G|}{|ker\varphi|}$, so $|\varphi(G)|$ is a divisor of $|G|$. Since $\varphi(G) \leq H$, Lagrange's Theorem also requires that $|\varphi(G)|$ be a divisor of $|H|$. However, since $|G|$ and $|H|$ are relatively prime, their only common divisor is 1, so $|\varphi(G)| = 1$, and thus the above equality derived from the First Isomorphism Theorem becomes $1 = \frac{|G|}{|\ker \varphi|}$, and so $|\ker \varphi| = |G|$. Since $\ker \varphi \subseteq G$ and the two sets have the same finite size, $\ker \varphi = G$.

4. *Determine, with proof, how many nonisomorphic Abelian groups there are of order 360.*

   Note that $360 = 2^3 3^2 5^1$. We may say with absolute certainty, from the Fundamental Theorem of Abelian Groups, that a group G of order 360 can be written as a direct product of three groups of order 2^3, 3^2, and 5. The third of these is uniquely determined to be Z_5; the second could be either Z_9 or $Z_3 \oplus Z_3$; and the first is one of Z_8, $Z_4 \oplus Z_2$, or $Z_2 \oplus Z_2 \oplus Z_2$. Thus, there are in fact 6 different Abelian groups of order 360.

5. *You are told that a group G is an Abelian group of order 16 and that there are elements a and b of G which both have order 4, and for which $a^2 \neq b^2$. Which isomorphism classes could contain this group?*

There are 5 partitions of the number 4, which correspond to 5 possible Abelian groups of order 16 by the Fundamental Theorem of Abelian Groups. So even without the condition on a and b, we can limit the candidates for group G to the following four: Z_{16}, $Z_8 \oplus Z_2$, $Z_4 \oplus Z_4$, $Z_4 \oplus Z_2 \oplus Z_2$, and $Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2$. We can consider each of these individually.

Case I: $G = Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2$. This group has no elements of order 4 and thus no such a and b exist.

Case II: $G = Z_8 \oplus Z_2$. (x, y) has order 4 only if at least one of x and y has order 4. No element of Z_2 has order 4, and 2 and 6 are the only elements of Z_8 of order 4. Thus a and b must be of the form $(2, x)$ and/or $(6, y)$. However $(2, x)^2 = (4, 0)$ and $(6, y)^2 = (4, 0)$, so a^2 and b^2 cannot be unequal.

Case III: $G = Z_4 \oplus Z_4$. The elements $a = (1, 0)$ and $b = (0, 1)$ satisfy the given conditions on a and b.

Case IV: $G = Z_4 \oplus Z_2 \oplus Z_2$. (x, y, z) has order 4 only if at least one of x, y, and z has order 4. No element of Z_2 has order 4, and 1 and 3 are the only elements of Z_4 of order 4. Thus a and b must be of the form $(1, y, z)$ and/or $(3, y', z')$. However $(1, y, z)^2 = (2, 0)$ and $(3, y', z')^2 = (2, 0)$, so a^2 and b^2 cannot be unequal.

Case V: $G = Z_{16}$. This group has exactly two elements of order 4: 4 and 12, but $2 \cdot 4 = 2 \cdot 12$, so this does not satisfy the necessary conditions for a and b.

Thus we see that G must be isomorphic to $Z_4 \oplus Z_4$.