1. **(15 points)** *Either prove the following conjecture, or furnish a counterexample.*

   Conjecture 1. *For any group G, subgroup H of G, and element k of G, the set $S = \{khk^{-1} : h \in H\}$ is a subgroup of G.*

   This is true because S will in fact be closed under multiplication and inversion.

   Note that any two elements of S will have the form kh_1k^{-1} and kh_2k^{-1} for $h_1, h_2 \in H$, and their product will be
   $$kh_1k^{-1}kh_2k^{-1} = k(h_1h_2)k^{-1}$$
   and, since H is a group and thus closed, $h_1h_2 \in H$, so $k(h_1h_2)k^{-1} \in S$; thus the product of two elements of S is in S.

   Likewise, for $khk^{-1} \in S$, its inverse is $(khk^{-1})^{-1} = (k^{-1})^{-1}h^{-1}k^{-1} = kh^{-1}k^{-1}$. Since H is a group, it is closed under inversion, so $h^{-1} \in H$ and thus $kh^{-1}k^{-1} \in S$; thus the inverse of any element of S is in S.

2. **(15 points)** *Let H_1 and H_2 be subgroups of a group G such that H_1 is not a subgroup of H_2 and H_2 is not a subgroup of H_1. Prove that the set $H_1 \cup H_2$ is not a group.*

   Since H_1 and H_2 are mutually not each other's subsets, then there is an element $a \in H_1$ such that $a \notin H_2$, and $b \in H_2$ such that $b \notin H_1$. Since H_1 and H_2 are groups, we can additionally note that, since groups are closed under the inversion operation, $a^{-1} \in H_1$ and $b^{-1} \in H_2$.

   Now note that both a and b are in $H_1 \cup H_2$. We shall show that $H_1 \cup H_2$ is not a group by demonstrating that $ab \notin H_1 \cup H_2$.

   If ab were in H_1, then since $a^{-1} \in H_1$, $a^{-1}ab \in H_1$; this is the same as asserting the clearly false claim that $b \in H_1$. Likewise, if ab were an element of H_2, then abb^{-1} would be in H_2 as well, leading to the contradictory result that $a \in H_2$. Thus, ab is in neither H_1 nor H_2, so $ab \notin H_1 \cup H_2$, and thus $H_1 \cup H_2$ is not closed under multiplication.

3. **(15 points)** *The following questions relate to the additive group modulo 30, known variously as either Z_{30} or $(\mathbb{Z}_{30}, +)$*

   (a) **(5 points)** *Which, if any, elements of this group have order 12? Explain your reasoning.*
   Since 30 is not divisible by 12, no element of the group has order 12.

   (b) **(5 points)** *How many subgroups does Z_{30} have, including itself and the trivial one-element subgroup?*
   By the Fundamental Theorem of Cyclic Groups, a cyclic group has exactly one subgroup of each order divisible by the order of the supergroup. Since 30 has 8 divisors, there are 8 subgroups of Z_30 (specifically one subgroup with each of the orders 1, 2, 3, 5, 6, 10, 15, and 30).

   (c) **(5 points)** *List all the generators of this group.*
   We want numbers less than and relatively prime to 30: those are 1, 7, 11, 13, 17, 19, 23, and 29.

4. **(20 points)** *Identify each of the two following algebraic structures as either a group or not as a group. Explain your reasoning and, if the structure in question is a group identify it as Abelian or non-Abelian.*

(a) *The set of subsets of* $\{1, 2, 3, 4, 5\}$ *under the operation* $\triangle$ *of symmetric difference, given by the rule* $S \triangle T = (S - T) \cup (T - S)$*, e.g.* $\{1, 2, 4\} \triangle \{2, 4, 5\} = \{1, 5\}$*.*

This operation is an Abelian group; note that symmetric differences of sets are sets, so we have closure; they exhibit symmetry with regard to operands, and are thus commutative. Associativity is a bit more difficult, but note that $R \triangle S \triangle T$ will contain exactly those numbers appearing in either one or three of the sets R, S, and T, regardless of the grouping. The identity element will be $\emptyset$, and each set is its own inverse, since $S \triangle S = \emptyset$.

(b) *The set of ordered pairs of integers under the operation* $*$ *defined by the rule* $(a, b) * (c, d) = (ac + ad, bd)$*.*

This operation is not a group, and one easy way to show it is by the nonexistence of a consistent identity: $(a, b) * (c, d) = (a, b)$ only when $(c, d) = 0, 1)$, but $(0, 1) * (a, b) \neq (a, b)$.

5. **(20 points)** *Answer the following questions about symmetric groups.*

(a) **(8 points)** *How many odd permutations are there in* S_5*? Explain your reasoning.*

S_5 contains 120 elements, of which exactly half are odd, so there are 60 odd permutations.

(b) **(6 points)** *What is the order of the following permutation?*

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 5 & 8 & 9 & 4 & 2 & 1 & 7 & 6 & 3 \end{bmatrix}$$

Written in cycle notation, it is $(15286)(39)(4)(7)$, whose order is, by Ruffini's theorem, the least common multiple of 5 and 2, which is 10.

(c) **(6 points)** *Is the permutation from the previous part odd or even? Explain why.*

We may write the 5-cycle as a product of 4 swaps, to express the above permutation as $(15)(52)(28)(86)(39)$, which, as a product of 5 swaps, is odd.

6. **(15 points)** *Find a subgroup* H *of the dihedral group* D_9 *such that* $|H| = 6$ *(note that* $|D_9| = 18$*). Demonstrate that* D_9 *does not contain an* Abelian *subgroup of order 6.*

One such subgroup is $\{e, r^3, r^6, f, fr^3, fr^6\}$.

To demonstrate nonexistence of an Abelian subgroup of order 6, let us note that $\{e, r, r^2, r^3, \ldots, r^8\}$ is a cyclic subgroup of order 9; since 6 is not a divisor of 9, this subgroup has no subgroup of order 6. Thus, in order to build a subgroup G of D_9 of order 6, we will need to include at least one flip. WLOG we could actually assume $f \in G$, since an appropriate automorphism of D_6 could map any flip to the flip called "f". Now, if G includes any pure rotations, it will be non-Abelian, since $r^k f = fr^{-k} \neq fr^k$ in a dihedral group of odd subscript (note that in a dihedral group D_{2k}, $r^k f$ does equal fr^k). Thus, G must consist *solely* of 5 flips and the identity; however, if G contains both the fundamental flip f and some other flip fr^k, then their product $f(fr^k)$, which equals r^k, must be an element of G.