1. **(15 points)** *Prove that, if G is a finite group of odd order, then the product of all the elements of G is the identity.*

   Proof. Since G has odd order, Lagrange's Theorem assures us every element of G has odd order; specifically, no element of G has order 2, and thus for every nonidentity $g \in G$, $g \neq g^{-1}$. Thus. the elements of G can be separated into three parts: $\{e\}$ and two parts which contain each other's inverse (we may do this algorithmically: for an arbitrary unclassified g, put g in S_1 and g^{-1} in S_2; repeat until all elements of G are classified). Then since each pair has identity product, the product as a whole is the identity. □

   Alternative proof. Let us give the collective product a name $x = \prod_{g \in G} g$. Then, since the invcerse is a bijective map from G to itself, we may also write $x = \prod_{g \in G} g^{-1}$. Then:

$$x^2 = \prod_{g \in G} g \prod_{g \in G} g^{-1} = \prod_{g \in G} (gg^{-1}) = \prod_{g \in G} e = e$$

   so since $x^2 = e$, the order of x divides 2. Since G has odd order, x cannot have order 2 by Lagrange's Theorem, so x has order 1 and thus $x = e$. □

2. **(20 points)** *Below, let $N \trianglelefteq G$ and $H \leq G$.*

   (a) **(15 points)** *Prove that $(N \cap H) \trianglelefteq H$.*

   Proof. We shall show that conjugation by an element of h maps an element of $N \cap H$ back onto itself. Let $n \in N \cap H$ and $h \in H$; we wish to show that $hnh^{-1} \in N \cap H$. Clearly, hnh^{-1} is a product of three elements of H and by multiplicative closure lies in H. Since $H \trianglelefteq G$, we know that conjugation by an element of G maps elements of N back to N; since $h \in H \leq G$ and $n \in N$, we may thus assert that the conjugate $hnh^{-1} \in N$. Since hnh^{-1} is in both N and H, it is in $N \cap H$. □

   Alternative proof. Since $N \trianglelefteq G$, because every normal subgroup is the kernel of some homomorphism, there is a homomorphism $\varphi : G \to \overline{G}$ (for some group $\overline{G}$) such that $\ker \varphi = N$. Now let us consider the restriction $\varphi|_H$ of φ to H; such a restriction will still satisfy the homomorphism criterion, so $\varphi|_H : H \to \overline{G}$ is a homomorphism. Note that $\varphi|_H(h) = \varphi(h)$ by the definition of a restriction, so the kernel of $\varphi|_H$ contains every element of H in $\ker \varphi$; i.e., $\ker \varphi|_H = H \cap \ker \varphi = H \cap N$. Since $\varphi|_H$ is a homomorphism, its kernel $H \cap N$ is normal in its domain H. □

   (b) **(5 points)** *Demonstrate by example (specific choices of G, N, and H) that it might* not *be the case that $(N \cap H) \trianglelefteq G$.*

   A nice trivial way to do this is to let $N = G = \{D_4\}$ and $H = \{e, f\}$, and then we have our old friend, the fact that $\{e, f\} \ntrianglelefteq D_4$ because $r^{-1}fr = fr^2$.

3. **(15 points)** *Answer the following questions about direct products.*

   (a) **(5 points)** *Is $Z_4 \oplus Z_3$ isomorphic to Z_{12}? Explain your reasoning.*

   Yes; note that 1 is an element of Z_4 of order 4 and 1 is an element of Z_3 of order 3, so the order of $(1,1)$ in $Z_4 \oplus Z_3$ is $\text{lcm}(4,3) = 12 = |Z_4 \oplus Z_3|$, so $(1,1)$ generates $Z_4 \oplus Z_3$; since $Z_4 \oplus Z_3$ is cyclic, it is isomorphic to Z_{12}.

(b) **(10 points)** *Groups G and H have orders 8 and 9 respectively. Prove that $G \oplus H$ contains an element of order divisible by 6. (Please note that G and H need not be cyclic or even Abelian.)*

By Lagrange's Theorem, elements of G have order 1, 2, 4, or 8; thus every non-identity element of G has order divisible by 2; let $g \in G$ be an element of even order. Likewise, elements of H have order 1, 3, or 9, so every non-identity element of H has order divisible by 3; let h be one such. The order of (g, h) is the least common multiple of $|g| = 2k$ and $|h| = 3\ell$, which must be divisible by 6.

4. **(15 points)** *For each group G and subgroup H listed below, identify H as either a normal subgroup or not a normal subgroup of G, and explain why:*

(a) **(5 points)** $G = D_{12} \oplus Z_2$, $H = \langle r \rangle \oplus Z_2$.

This is normal: Note that H has index 2 in G, and thus its only coset is $G - H$. We know thus that $gH = \begin{cases} H \text{ if } g \in H \\ G - H \text{ if } g \notin H \end{cases}$; the same cases describe the possible values of Hg, and so $gH = Hg$.

(b) **(5 points)** $G = S_4$, $H = \{e, (12), (34), (12)(34)\}$.

This is not normal: $(13)^{-1}(12)(13) = (23) \notin H$.

(c) **(5 points)** $G = Z$, $H = \{\ldots, -6, -3, 0, 3, 6, \ldots\}$.

This is trivially normal; G is Abelian, and every subgroup of an Abelian group is normal.

5. **(20 points)** *Answer the following questions about homomorphisms.*

(a) **(5 points)** *Identify the kernel and image of the homomorphism from D_4 to $Z_2 \oplus Z_4$ (the infinite cyclic group) given by the rules $\varphi(r) = (1, 0)$ and $\varphi(f) = (0, 2)$.*

The image of this homomorphism is the group $Z_2 \oplus \{0, 2\}$. THe kernel is $\{e, r^2\}$.

(b) **(10 points)** *Show that no homomorphism from D_4 to Z_4 is surjective.*

If ϕ is surjective, then 1 must be the image of an element of order divisible by 4, which is either r or r^{-1}; since an automorphism of D_4 maps one to the other, we may wlog assume $\phi(r) = 1$. Then

$$1 = \phi(r) = \phi(ffr) = \phi(f) + \phi(r) + \phi(f) = \phi(frf) = \phi(r^{-1}) = 3$$

which is impossible.

(c) **(5 points)** *Using the definition of φ from part (a), what well-known group is $G/\ker\varphi$ isomorphic to?*

By the First Isomorphism Theorem, it is isomorphic to the image of φ, which is $Z_2 \oplus \{0, 2\}$, which is isomorphic to $Z_2 \oplus Z_2$ (clearly demonstrated by dividing all elements by 2), also known as the Klein 4-group.

6. **(15 points)** *Prove that if $N \trianglelefteq G$ and $M \trianglelefteq G$, then $N \cap M \trianglelefteq G$.*

Proof. Let $k \in N \cap M$ and $g \in G$; we wish to show that $gkg^{-1} \in N \cap M$. Since $k \in N$ and $N \trianglelefteq G$, $gkg^{-1} \in N$. Likewise, since $k \in M$ and $M \trianglelefteq G$, $gkg^{-1} \in M$. Thus $gkg^{-1} \in N \cap M$. □