1. **(20 points)** For each of the following sets and operations thereon, check whether they form a group; show your work, and for each that is not a group, say why not.

   (a) The set $\mathbb{R} \times \mathbb{R}$ under the multiplication operation $(a, b)(c, d) = (ad, bc)$.

   (b) The set $\{\ldots, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 1, 2, 4, 8, 16, \ldots\}$ under the operation of standard real-number multiplication.

   (c) The set of all sets of positive rational numbers under the rule that $S \cdot T = \{st : s \in S, t \in T\}$. (For instance, $\{\frac{1}{2}, \frac{1}{3}\} \cdot \{\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \ldots\} = \{\frac{1}{15}, \frac{1}{10}, \frac{2}{15}, \frac{1}{5}, \frac{3}{10}, \frac{4}{15}, \frac{2}{10}, \ldots\}$.)

2. **(30 points)** *Prove or disprove* the following statements about subgroups.

   (a) For any finite group G, the set $\{g \in G : g^2 = e\}$ is a subgroup of G.

   (b) For any group G and element x thereof, the set $\{g \in G : gxg^{-1} = x\}$ is a subgroup of G.

   (c) For any integers m and n, D_{mn} has a $2m$-element subgroup.

3. **(20 points)** Consider elements a and b ate elements of a group G such that the order of a is m and the order of b is n.

   - Prove that if $ab = ba$, then the order of ab is divisible by mn.
   - Demonstrate that when $ab \neq ba$, then the order of ab might not be divisible by mn.

4. **(20 points)** Answer the following questions.

   (a) **(10 points)** Identify a cyclic group Z_n which has exactly 6 subgroups, including Z_n itself and the trivial subgroup $\{e\}$.

   (b) **(10 points)** Suppose $H \leq G$ and $|H| = 10$. a is an element of G such that $a^6 \in H$. What are the possible values of $|a|$ and why?

5. **(20 points)** Let H and K be subgroups of G such that for some $a, b \in G$, it is the case that $aH \subseteq bK$. Prove that $H \subseteq K$.

6. **(25 points)** Let $N \leq H \leq G$, and furthermore $N \trianglelefteq G$. Prove that H/N is normal in G/N if and only if H is normal in G.

7. **(15 points)** Prove that for any homomorphism $\varphi : G \to H$ and element a of G, the order of $\varphi(a)$ divides the order of a.