1. **(15 points)** *The following questions relate to the additive group modulo 18, known variously as either Z_{18} or $(\mathbb{Z}_{18}, +)$*

   (a) **(5 points)** *Which, if any, elements of this group have order 6? Explain your reasoning.*
   Since Z_{18} is cyclic with generator 1, any number k with $\gcd(k, 18) = \frac{18}{6} = 3$ will work. These numbers are 3 and 15.

   (b) **(5 points)** *What are the orders of the following elements of the group: 9, 10, and 11?*
   Since $\gcd(9, 18) = 9$, $\gcd(10, 18) = 2$, and $\gcd(11, 18) = 1$, these have orders of $\frac{18}{9} = 2$, $\frac{18}{2} = 9$, and $\frac{18}{1} = 18$ respectively.

   (c) **(5 points)** *List all the generators of this group.*
   We want numbers less than and relatively prime to 18: those are 1, 5, 7, 11, 13, and 17.

2. **(15 points)** *Describe (either in words or with a Cayley table) a* noncyclic *group of order 4. Justify your assertion that this group is noncyclic.*

   Here is a Cayley table:

	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

   We could equivalently describe it as the group on oderered pairs modulo 2 by pairwise addition, or reference it by name as the Klein 4-group, Or describe it as a 4-element Abelian group where each $x^2 = e$ and where the product of any two non-identity elements is the third non-identity.

   We can demonstrate this group is noncyclic sicne the group has order 4 and no element has order 4; thus no element generates the whole group.

3. **(15 points)** *Let a and b be elements of a group. If the orders of a and b are relatively prime, prove that $\langle a\rangle \cap \langle b\rangle = \{e\}$.*

   Note that $e \in \langle a\rangle \cap \langle b\rangle$ trivially. Now we wish to show that there is no *other* element of $\langle a\rangle \cap \langle b\rangle$.

   Let us call the orders of a and b by the names k and ℓ respectively (so $\gcd(k, \ell) = 1$, $a^k = e$, and $b^\ell = e$). Let us consider some element $g \in \langle a\rangle \cap \langle b\rangle$. Let g have order x. Since $\langle a\rangle$ is cyclic and $g \in \langle a\rangle$, $x \mid k$. Likewise, since $\langle b\rangle$ is cyclic and $g \in \langle b\rangle$, $x \mid \ell$. Since x is a positive common divisor of relatively prime k and ℓ, $x = 1$, and since g has order x, g must be the identity.

4. **(20 points)** *Answer the following questions about symmetric groups.*

   (a) **(10 points)** *What is the smallest value of n such that S_n contains an element of order 10? Explain your reasoning.*
   S_7 contains an element of order 10, for example $(12345)(67)$. Since the order of an element is the least common multiple of the orders of its constituent cycles, a permutation can only have order 10 if it consists of at least one cycle of order divisible by 5 and at least one cycle of even order; these could be the same, in which case the permutation would need a cycle of length divisible by 10, or they could be different, in which case the permutation would need to be the product of disjoint cycles of length at least 5 and 2, which is only possible if there are at least $5 + 2 = 7$ objects on which the permutation is performed.

(b) **(5 points)** *What is the order of the following permutation?*

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 2 & 7 & 1 & 3 & 6 & 5 \end{bmatrix}$$

Written in cycle notation, it is (14)(375), whose order is, by Ruffini's theorem, the least common multiple of 2 and 3, which is 6.

(c) **(5 points)** *Is the permutation from the previous part odd or even? Explain why.*

We may write the 3-cycle as a product of 2 swaps, to express the above permutation as (14)(37)(75), which, as a product of 3 swaps, is even.

5. **(20 points)** *Identify each of the two following algebraic structures as either a group or not as a group. Explain your reasoning and, if the structure in question is a group identify it as Abelian or non-Abelian.*

(a) *The set of real numbers under the operation $a \star b = ab + b$ (e.g. $6 \star (-2) = -10$).*

The operation is nonassociative: $2 \star (3 \star 4) = 2 \star 16 = 48$, while $(2 \star 3) \star 4 = 9 \star 4 = 40$. Thus, this is not a group.

(b) *The set of positive real numbers under the operation of multiplication.*

This operation is an Abelian group: products are associative and commutative, and the product of two positive real numbers is a positive real number, the identity is 1, and every x has $\frac{1}{x}$ as an inverse.

6. **(15 points)** *Either prove the following conjecture, or furnish a counterexample*

Conjecture 1. *For any group G, subgroup H of G, and element k of G, the set $\{kh : h \in H\}$ is a subgroup of G.*

This is not a true conjecture, and it is easy to demonstrate it: let $G = D_4$, $H = \{e\}$, and $k = r$. Then $\{kh : h \in H\}$ is the set $\{r\}$, which is not a subgroup of G, since it is not closed under multiplication ($r \cdot r \notin \{r\}$).